# The Impact of Service Level Constraints in Deterministic Lot Sizing with Backlogging

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**Abstract**

This paper studies the impact of service level constraints in the context of the capacitated and uncapacitated lot sizing problems with deterministic demand and backlogging over a discrete and finite time horizon. Because the standard formulation of these problems cannot distinguish between a backlog and a backorder, we use a reformulation to explicitly make this distinction. We then formulate several service level constraints adapted from the stochastic inventory control literature and introduce new ones. We analyze the impact of these constraints on the structure of the optimal solutions in terms of the number of backorders and total backlog, and on the performance of these solutions in terms of various service level indicators (proportion of periods with no backlog, proportion of backorders over total demand, maximum delay, average unit waiting time for an item). We also consider the introduction of various types of backordering costs (both fixed and duration dependent) and study their impact. Finally, we analyze the impact of a first-in first-out order management policy and show that it helps improve most solutions.

*Keywords:* Production planning and control; lot sizing; deterministic demand; service levels; backlogging; mixed integer programming formulations.

## 1. Introduction

The deterministic lot sizing literature usually ignores the issues of backlogging and service levels because of an implicit assumption that shortages should not occur when demand is known in advance. In practice, however, it often makes sense to plan backorders when the demand is highly dynamic or setup costs are high. In the presence of capacity constraints, it may also be profitable to backorder the demand of a less profitable item so that the demand of a more profitable item is filled on time. The starting point of our research is the recent article of Gade and Küçüyavuz [1], which addresses the uncapacitated lot sizing problem with service level constraints in the deterministic case. Two service levels are considered by the authors: one that limits the number of periods with a backlog and one that ensures that a given percentage of the demand is met on time. They focus on solution methods and propose a polynomial time algorithm for an uncapacitated lot sizing problem with a limit on the number of periods with a backlog. In this paper, we propose several other service level constraints and we highlight the fact that different constraints may lead to completely different solutions. This important issue for the industry is seldom taken into account despite the fact that it directly affects the quality of service. We further introduce costs related to backorders, whereas in the literature the costs are traditionally related to the backlog. Finally, we also propose a new model that imposes an order management policy.

We consider the deterministic version of both the capacitated lot sizing problem (CLSP) and the uncapacitated lot sizing problem (ULSP). The basic lot sizing problem determines the optimal timing and size of the lots that must be produced in order to meet a demand that varies dynamically in a discrete and finite time horizon. The objective is to minimize the sum of the setup costs, inventory holding costs and production costs. The basic MIP formulation presented by Pochet and Wolsey [2] is based on three sets of variables: $x_{it}$ represents the production of item $i$ in period $t$, $s_{it}$ is the inventory of item $i$ at the end of period $t$, and $y_{it}$ is a boolean setup variable taking value 1 if and only if there is a setup for item $i$ in period $t$. We denote by $T$ the set of time periods in the planning horizon and by $I$ the set of items. Demand for item $i$ in period $t$ is $d_{it}$ and the cumulative demand for item $i$ from period $t$ to period $q$ $(t < q)$ is $D_{itq}$. As mentioned previously, there are also three types of costs: $vc_{it}$ is the unit production cost for item $i$ in period $t$, $hc_{it}$ is the unit holding cost for item $i$ in period $t$, and $sc_{it}$ is the setup cost for item $i$ in period $t$. The ULSP can be formulated as follows:

$$\text{Min} \sum_{i=1}^{|I|} \sum_{t=1}^{|T|} (sc_{it}y_{it} + vc_{it}x_{it} + hc_{it}s_{it}) \tag{1}$$

$$s_{i,t-1} + x_{it} = d_{it} + s_{it} \qquad \forall i \in I, t \in T \tag{2}$$

$$x_{it} \leq D_{it|T|}y_{it} \qquad \forall i \in I, t \in T \tag{3}$$

$$x_{it}, s_{it} \geq 0, s_{i0} = 0 \qquad \forall i \in I, t \in T \tag{4}$$

$$y_{it} \in \{0, 1\} \qquad \forall i \in I, t \in T. \tag{5}$$

Constraints (2) are inventory balance and demand satisfaction constraints whereas constraints (3) are the setup constraints which force the binary variable $y_{it}$ to take value 1 if any production of item $i$ occurs in period $t$. The MIP formulation can be extended to include the possibility of planned backlogs (Pochet and Wolsey [3], Zangwill [4]). Let $b_{it}$ be the amount of backlog for item $i$ at the end of period $t$ and let $bc_{it}$ be the unit backlogging cost for item $i$ at the end of period $t$. The formulation can be adapted as follows:

$$\text{Min} \sum_{i=1}^{|I|} \sum_{t=1}^{|T|} (sc_{it}y_{it} + vc_{it}x_{it} + hc_{it}s_{it} + bc_{it}b_{it}) \tag{6}$$

$$s_{i,t-1} + x_{it} + b_{it} = d_{it} + s_{it} + b_{i,t-1} \qquad \forall i \in I, t \in T \tag{7}$$

$$x_{it} \leq D_{i1|T|}y_{it} \qquad \forall i \in I, t \in T \tag{8}$$

$$x_{it}, s_{it}, b_{it} \geq 0 \qquad \forall i \in I, t \in T \tag{9}$$

$$s_{i0} = b_{iT} = 0 \qquad \forall i \in I \tag{10}$$

$$y_{it} \in \{0, 1\} \qquad \forall i \in I, t \in T. \tag{11}$$

The inventory balance equations (7) now allow backlogging. In the setup forcing constraints (8), the maximum amount that can be produced is the total demand over the whole horizon. This MIP formulation, however, has a major drawback: it provides information about backlogs, but not about backorders.

The backorder in period $t$ refers to the quantity of unmet demand in that period whereas the backlog in period $t$ is the number of backorders in period 1 to $t$ that have not been filled by the end of period $t$ (Gade et al. [1], Helber et al. [5]). Figure 1 illustrates this deficiency in the MIP formulation. Note that in Figure 1, the item index has been removed from variables and parameters for simplicity. Nodes represent periods and edges represent demand parameters ($d_t$), stock variables ($s_t$), production variables ($x_t$) and backlog variables ($b_t$).

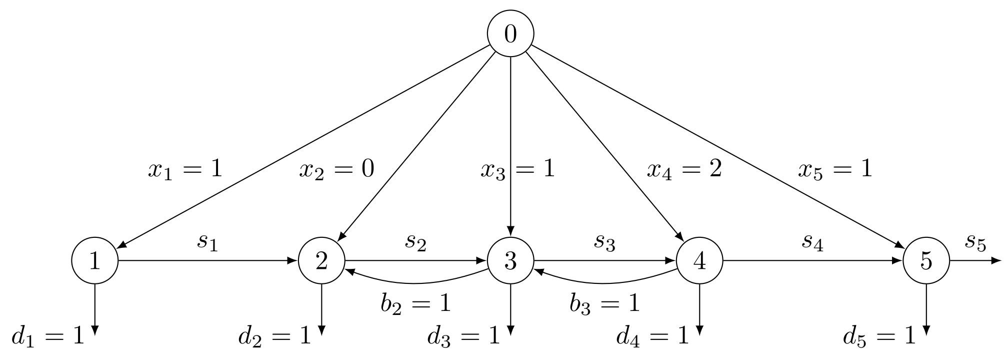


Figure 1: Graphical representation of the MIP model with backlog

Assume that each period has a demand of one unit. Given the production values, there is no inventory carried over. One can see that there is a backlog of one unit at the end of period 2 ($b_2 = 1$) and a backlog of one unit at the end of period 3 ($b_3 = 1$). However, this solution can represent two very different situations. In the first case there is one backorder in period 2. This order is satisfied with a delay of two periods, and the unit demand in period 3 is directly satisfied from the production in period 3. In the second case there is a backorder in period 2 which is satisfied with a delay of one period, and another backorder in period 3 which is also satisfied with a delay of one period. In both cases, this leads to a backlog of one unit in period 2 and one unit in period 3. The standard formulation cannot distinguish between these two cases, although they (possibly) lead to different service levels. The fundamental reason is that the standard formulation does not have a variable for the backorder.

To tackle this problem, it is convenient to use a reformulation. Pochet and Wolsey [2] indicate that the facility location reformulation proposed by Krarup and Bilde [6] can be extended to include backlog and a similar reformulation is used in Gade et al. [1]. Here, $z_{ikt}$ denotes the percentage of the demand of item $i$ in period $t$ that is satisfied by production in period $k$. As backlogging is allowed, one can have $k > t$. We associate to these variables the following costs:

$$c_{ikt} = \begin{cases} \left(vc_{ik} + \sum_{l=k}^{t-1} hc_{il}\right) d_{it} & \text{if } k \leq t \\ \left(vc_{ik} + \sum_{l=t}^{k-1} bc_{il}\right) d_{it} & \text{if } k > t. \end{cases}$$

The model is then as follows:

$$\text{Min} \sum_{i=1}^{|I|} \sum_{t=1}^{|T|} \left( sc_{it} y_{it} + \sum_{k=1}^{|T|} c_{ikt} z_{ikt} \right) \tag{12}$$

$$\sum_{k=1}^{|T|} z_{ikt} = 1 \qquad \forall i \in I, t \in T \tag{13}$$

$$z_{itk} \leq y_{it} \qquad \forall i \in I, k, t \in T \tag{14}$$

$$z_{ikt} \geq 0 \qquad \forall i \in I, k, t \in T \tag{15}$$

$$y_{it} \in \{0, 1\} \qquad \forall i \in I, t \in T. \tag{16}$$

Constraints (13) guarantee the satisfaction of the demand whereas (14) are the setup constraints. This variable redefinition provides more information and allows us to distinguish between a backlog and a backorder (Gade et al. [1]). With this variable redefinition, one can express the backlog of item $i$ in period $t$ as follows:

$$b_{it} = \sum_{j=1}^{t} \sum_{k=t+1}^{|T|} d_{ij} z_{ikj}. \tag{17}$$

The backorder of item $i$ in period $t$, represented by variable $bo_{it}$, can be calculated as:

$$bo_{it} = \sum_{k=t+1}^{|T|} d_{it} z_{ikt}. \tag{18}$$

Using this richer formulation, it is now possible to add service levels constraints related to both the backlogs and the backorders. A service level constraint ensures that the solution will satisfy a certain measure of quality of service related to the demand satisfaction.

Our paper makes several contributions. First, we define and model several backlog and backorder related service levels, each dealing with a different quality aspect. For each of these service levels, we then report the results of extensive computational experiments for the ULSP and CLSP. Some experiments for the ULSP are performed with an order management policy so as to assess the impact of this policy on the results compared to the ULSP. Other experiments are performed with both fixed and time-dependent backorder costs. To the best of our knowledge, the idea of using backlog costs has been discussed in several papers but with little experimentation because of the difficulty of finding the appropriate cost. We overcome this difficulty by taking a wide range of possible values for these costs. It is worth mentioning that we will consider only backorder costs and not backlog costs. Indeed, we consider that only backorders actually lead to additional costs and not backlog. The time dependent backorder cost, however, allows us to take the time dimension (i.e. the duration of the backorder) into account. With all the results, we show that different service level constraints lead to completely different solutions in terms of objective function value, structure and performance indicators, which we denote as Key Performance Indicators (KPIs). This highlights the importance of properly choosing and setting a service level constraint in practice.

The remainder of this paper is organized as follows. First, we survey the work related to our study in Section 2. We then explain how various service level constraints can be formulated and added to the facility location reformulation in Section 3 and we consider backorder costs in Section 4. We also propose the introduction of an order management policy in Section 5. Finally, Section 6 presents computational results to analyze the impact of the different service levels on the optimal solutions. This analysis is performed on the ULSP, the ULSP with stationnary costs, the ULSP with backorder cost, the ULSP with a

First-In-First-Out (FIFO) order management policy and the CLSP.

## 2. Literature review

### 2.1. Lot sizing and backlogging

For the deterministic lot sizing problem, there are numerous papers that study extensions of the basic problem: setup times, time windows, multiple levels, perishability, etc. There are also several papers describing special problems coming from the industry with a focus on solving realistic problems by using heuristics or metaheurisitcs. For a review of various solution approaches used in lot sizing problems, we refer to Jans et al. [7] and for a review of modeling aspects, to Jans et al. [8]. Reviews of the different solution approaches used in many papers can also be found in Buschkhühl et al. [9] and Robinson et al. [10]. An excellent detailed discussion of lot sizing models can be found in Pochet and Wolsey [2].

Zangwill [4] was the first to introduce backlog in deterministic lot sizing models. In [11], the same author uses a different formulation for the ULSP with backlog and obtains a network formulation with one source node. The network obtained is the same as shown in Figure 1. He considers concave costs both for holding and backlog costs. He characterizes properties for the optimal solution and, based on these, he develops dynamic programming algorithms. In the same line of research, several papers study the convex hull of solutions and develop valid inequalities in order to solve the ULSP problem with backlog as a linear program. The main papers are those of Pochet and Wolsey [3], [12] and some improvements can be found in Küçükyavuz and Pochet [13]. From another perspective, Federgruen and Tzur [14] develop algorithms to solve the standard ULSP with backlogging with a maximum complexity of $O(n \log n)$, where $n$ denotes the lenght of the time horizon.

For the CLSP with backlog, Constantino [15] uses cutting planes in a branch-and-cut algorithm to solve some real life instances and overcome the fact that bounds obtained from linear programming relaxations are weak. Similar work is done by Van Vyve [16]. In this paper, the author extends formulations provided by Pochet and Wolsey [12] to the case with constant capacity constraints.

Backlogging has also been considered in various extensions of the standard lot sizing problem, such as capacitated multi-level lot sizing in Wu et al. [17], lot sizing with production time windows in Absi et al. [18], an industrial extension of the discrete lot sizing and scheduling problem in Jans and Degraeve [19], and two-level serial lot sizing with cargo capacity in Solyali et al. [20].

### *2.2. Service levels*

Several service level constraints have been proposed, mainly in the context of stochastic inventory management problems. For an overview of stochastic inventory models, we refer to Zipkin [21]. From a practice point of view, Ronen [22] provides an overview of different service level measures in a stochastic setting and makes links between these measures from the point of view of product availability during lead time.

A popular service level constraint is the cycle service level constraint. In a stochastic setting, it denotes the probability that no stockout occurs during a production or procurement cycle [23]. In a discrete time model, the cycle may be replaced by a planning period as in Helber et al. [5]. This service level measure often appears when stockouts are rather costly and independent of the amount of unsatisfied demand. In Tunc et al. [24], this service level is used with an application of a variant of the "static uncertainty" strategy of Bookbinder and Tan [25]. This static uncertainty strategy imposes that the values of all decision variables are set at the beginning of the time horizon. The variant proposed in Tunc et al. [24] is called the static-dynamic uncertainty strategy. The authors propose a reformulation of the stochastic problem to obtain a better bound for the linear relaxation in a short time. This reformulation, based on binary variables that indicate if two periods are successive replenishment periods, is a deterministic equivalent of the MIP model.

The most common service level constraint is the fill rate which represents the fraction of the demand that is satisfied from inventory. In a stochastic setting, this level is defined as the ratio between the expected demand that is filled from stock during a replenishment cycle and the associated lot size. This measure is very often used in industry to assess the performance of inventory management systems in terms of quality of service. The fill rate constraint is present in Tempelmeier and Herpers [26], where the CLSP is studied with random demand over a finite discrete time horizon and unfilled demands are backordered. As demand is stochastic, a fill rate constraint is imposed. The authors present a formulation of the stochastic dynamic CLSP that implements the static uncertainty strategy. This fill rate service level constraint has also been considered for the CLSP under random demand in Tempelmeier [27]. In [27], the author uses a method that combines a column generation process and the $ABC_\beta$ heuristic described in Tempelmeier and Herpers [26] to solve the LP-relaxation of the CLSP. The author obtained better results than the use of $ABC_\beta$ heuristic alone.

The authors of [26] propose a heuristic solution procedure whose result is an a priori production plan for the entire horizon. The fill rate constraint is imposed for each item separately and the fill rate is computed as the complement of the ratio between the expected cumulative backorders since the last production period and the expected cumulative demand since the last production period. The fill rate constraint is also considered in Tempelmeier and Herpers [28], where the same model is presented for the stochastic case of the ULSP. A “static uncertainty” strategy is then used with several heuristics to make a comparison with exact solution approaches. It turns out that the Silver-Meal rule gives the best results for high fill rates with respect to the increase in the cost compared to the exact method. This Silver-Meal rule was challenged in Gaafar et al. [29] where the authors use a neural network instead of traditional MIP models or heuristics to solve the traditional lot sizing problem.

Two other measures are considered in Helber et al. [5], besides the fill rate and cycle service level. These are called the gamma and delta service levels. The gamma service level is the ratio between the expected backlog in period $t$ and the expected demand in period $t$. It reflects the magnitude of the backlog and, to some extent, the waiting time. Its drawback is that it can be undefined in the case of an expected demand of zero. Finally, the delta service level takes into account the waiting time of customers and the size of backorders. It is computed as the ratio between the total expected backlog and the total maximum expected backlog (which equals the total expected cumulative demand if nothing is produced until the considered period). The authors make a clear link between this measure and the waiting time of customers and then apply this constraint to their stochastic model for the CLSP. The authors explain that they chose this delta service level over the fill rate because it allows them to take into account the waiting time of customers and it also limits backlog. They apply this constraint to a non-linear stochastic version of the CLSP. As the problem is not linear, the authors develop two models to approximate it, which can both be solved with standard methods for MIP problems. The approximations are made thanks to a scenario approach (they use a set of demand scenarios and random variables are replaced by their realizations in the scenario) or with the use of piecewise linear approximations to replace expected backlog and physical inventory. The authors then use a fix-and-optimize algorithm to solve the two approximation models. The results show that the piecewise linear approximation performs well and that the scenario approach is more flexible in dealing with probabilistic dependencies of the demand.

Few papers have considered service levels in a deterministic lot sizing setting with planned backorders. Zangwill [4] is an exception: he imposes a limit on the number of periods a demand is delayed, which can be interpreted as a service level. More recently, the article of Gade and Kuçüyavuz [1] has introduced service levels for the deterministic ULSP with backlog. The authors consider an adapted version of the cycle service level (called alpha service level) in which the number of periods with a backlog is limited. They also consider fill rate service level constraints (called beta service level constraints).

## 3. Modelling service levels

As explained in the previous section, there are two main types of service levels considered in the stochastic lot sizing literature. To the best of our knowledge, these service level constraints are usually used in two ways when there are multiple products: globally for the cycle service level (i.e. considering all products jointly as in Helber et al. [5], Silver [23] and Tunc et al. [24]) and individually for the fill rate (i.e. considering products one by one as in Tempelmeier and Herpers [26], [28]). One contribution of this paper is the introduction of service level constraints at an individual level, item by item, period by period and globally. In the following sections, we define the various service levels that we consider in this study.

### *3.1. Period service level*

In the stochastic inventory literature, the cycle service level refers to the probability of having no stockout during a replenishment cycle. In our case, for both the deterministic ULSP and CLSP with a finite time horizon, we define the Period Service Level (PSL) as the percentage of time periods in which no backorder occurs. To count this number of time periods, we introduce a new binary variable $u_{it}$ taking the value 1 if and only if a backorder occurs for item $i$ in period $t$. It is defined as follows:

$$u_{it} \geq \sum_{k=t+1}^{|T|} z_{ikt} \qquad \forall i \in I, t \in T, \tag{19}$$

$$u_{it} \in \{0,1\} \qquad \forall i \in I, t \in T. \tag{20}$$

As mentioned above, we want to distinguish two cases: one case with a global constraint and one case with one constraint per item. To express the global constraint, we introduce a new binary variable $u'_t$ taking value 1 if and only if a backorder occurs in period $t$ for at least one item. These new variables are set as follows:

$$u'_t \geq u_{it} \qquad \forall i \in I, \forall t \in T \tag{21}$$

$$u'_t \in \{0,1\} \qquad \forall t \in T. \tag{22}$$

Using parameter $\alpha$ as the service level parameter, the Product Period Service Level constraint can then be written as follows:

$$1 - \frac{\sum_{t=1}^{|T|} u_{it}}{|T|} \geq \alpha \qquad \forall i \in I. \tag{23}$$

We call the resulting problem obtained by adding (23) to (12)-(16) the Product Period Service Level problem (PSL-P). In the global case, we add the following constraint:

$$1 - \frac{\sum_{t=1}^{|T|} u'_{t}}{|T|} \geq \alpha. \tag{24}$$

We denote this constraint as the Global Period Service Level constraint, which leads to the Global Period Service Level problem (PSL-G) if we add (24) to (12)-(16). It is easy to see that constraints (24) imply constraints (23).

### *3.2. Fill rate*

The fill rate, also called beta service level in the literature, refers to the proportion $\beta$ of demand filled directly from inventory over the total cumulative demand. In our case, it limits the amount of backorders or, in other words, the proportion of items not delivered directly from stock. For this well known measure, we also distinguish several cases. In the first case, we want to limit the backorders for each item and each time period. We thus impose Individual Fill Rate constraints:

$$1 - \frac{\sum_{k=t+1}^{|T|} z_{ikt} d_{it}}{d_{it}} \geq \beta \qquad \forall i \in I, t \in T. \tag{25}$$

It is of course possible to remove $d_{it}$ from both the numerator and the denominator in the second term of the left hand side of inequality (25). We call the resulting problem obtained by adding (25) to (12)-(16) the Individual Fill Rate problem (FR-I). In the second case, we limit the total backorders over all items for each time period and add Temporal Fill Rate constraints:

$$1 - \frac{\sum_{i=1}^{|I|} \sum_{k=t+1}^{|T|} z_{ikt} d_{it}}{\sum_{i=1}^{|I|} d_{it}} \geq \beta \qquad \forall t \in T. \tag{26}$$

We call the resulting problem obtained by adding (26) to (12)-(16) the Temporal Fill Rate problem (FR-T). In the third case, we limit the backorders over all time periods for each item and impose Product Fill Rate constraints. If we denote by $D_i$ the cumulative demand from the beginning until the end of the time horizon for item $i$, we have:

$$1 - \frac{\sum_{t=1}^{|T|-1} \sum_{k=t+1}^{|T|} z_{ikt} d_{it}}{D_i} \geq \beta \qquad \forall i \in I. \tag{27}$$

We call the resulting problem obtained by adding (27) to (12)-(16) the Product Fill Rate problem (FR-P). In the last case, which corresponds to the type II service level used in Gade et al. [1], we impose the constraint globally for all the items over the whole horizon and impose a Global Fill Rate constraint:

$$1 - \frac{\sum_{i=1}^{|I|} \sum_{t=1}^{|T|-1} \sum_{k=t+1}^{|T|} z_{ikt} d_{it}}{\sum_{i=1}^{|I|} D_i} \geq \beta. \tag{28}$$

We call the resulting problem obtained by adding (28) to (12)-(16) the Individual Fill Rate problem (FR-G). The different relationships between the interpretations of the Fill Rate are presented in Table 1.

| FR | (25) | (26) | (27) | (28) |
|---|---|---|---|---|
| (25) | | ⇒ | ⇒ | ⇒ |
| (26) | X | | X | ⇒ |
| (27) | X | X | | ⇒ |
| (28) | X | X | X | |

Table 1: Relationships between the interpretations of the Fill Rate constraints

### *3.3. Delta service level*

The following two service levels are motivated by the need to take backlog into account directly. The backlog can be seen as the depth of the unserved demand. Again, we have an individual case and a period by period case. The delta service level was first introduced by Helber et al. in [5].

We can introduce the following constraints, depending on the type of service level we want to take into account:

$$b_{it} \leq (1-\delta) D_{i1t} \qquad \forall i \in I, t \in T \tag{29}$$

$$\sum_{i=1}^{|I|} b_{it} \leq (1-\delta) \sum_{i=1}^{|I|} D_{i1t} \qquad \forall t \in T. \tag{30}$$

We call (29) Individual Backlog Depth constraints, which result in the Individual Backlog Depth problem (BD-I) by adding them to (12)-(16). We call (30) Temporal Backlog Depth constraints, which result in the Individual Backlog Depth problem (BD-I) by adding them to (12)-(16). In Helber et al. [5], both cases are presented but only the temporal constraint is used in the given stochastic model. One can observe that constraints (29) imply constraints (30). Using equality (17), we can easily rewrite these two previous constraints in terms of the $z_{ikt}$ variables.

### *3.4. Lambda service level*

We introduce a new service level which we call the lambda service level. This measure is inspired from Zangwill [4], who argues that customers should not wait for their goods more than a given number ($\lambda_{max}$) of periods. He imposes this service level in terms of inventory variables. To impose this measure in our problem, we proceed differently and discard the variables that allow a longer delay:

$$z_{ikt} = 0 \qquad \forall i \in I, (k,t) \in \{(k,t) \in T \times T | k - t > \lambda_{max}\}. \tag{31}$$

We call the resulting problem obtained by adding (31) to (12)-(16) the Product Maximum Delay problem (MD-P).

In a similar spirit, one may want to take into account the average waiting time of each unit of demand and impose the following constraint:

$$\frac{\sum_{i=1}^{|I|} \sum_{t=1}^{|T|} \sum_{k=t}^{|T|} z_{ikt}(k-t) d_{it}}{\sum_{i=1}^{|I|} \sum_{t=1}^{|T|} d_{it}} \leq \lambda. \tag{32}$$

This constraint limits the global average waiting time per unit of demand. We call the resulting problem obtained by adding (32) to (12)-(16) the Global Average Delay problem (AD-G).

## 4. Backorder costs

In the past research on backlogging in lot sizing problems, the traditional approach has been to include additional backlog costs in the objective function so as to control the frequency and importance of backlogging and backordering, instead of imposing a service level. In the dynamic programming approaches proposed by Zangwill [4], [11], the backlog cost can be a concave function of the amount backlogged. In the traditional MIP formulations, a unit backlog cost, which is multiplied by the backlog variable $b_{it}$ is added (Pochet and Wolsey [2]). Only a few papers have considered other types of costs. Gade et al. [1] mention that a fixed backlog cost could be added. Hsu and Lowe [30] consider the case where the backlog cost may increase in a non-linear way with the waiting time. The backlog cost hence depends on the time of production and the time of the demand. The authors present polynomial dynamic programming algorithms for several special cases. Using our proposed reformulation (12)-(16), such a delay-dependent backorder cost can easily be incorporated for each pair of time periods $(k, t)$ for which $k > t$, where $k$ represents the production period and $t$ represents the demand period. Let $bdc_{ikt}$ be the unit delay-dependent cost per backorder, which is produced in period $k$ to satisfy a demand in an earlier period $t$. The cost related to a $z_{ikt}$ variable is then defined as follows:

$$c_{ikt} = \begin{cases} \left(vc_{ik} + \sum_{l=k}^{t-1} hc_{il}\right) d_{it} & \text{if } k \leq t \\ (vc_{ik} + blc_{ikt})\, d_{it} & \text{if } k > t. \end{cases}$$

Note that in the traditional formulation where the cost does not depend on the length of the delay, the value of $blc_{ikt}$ is equal to $\sum_{l=t}^{k-1} bc_{il}$. To the best of our knowledge, no other models, apart from the one proposed in [31], have been proposed that impose a cost on the backorders. A fixed backorder cost, $boc_{it}$, can also be introduced, which will be associated with our previously defined decision variable $u_{it}$. The fixed backorder cost is charged whenever there is some backorder in a period, independent of the amount of backorders in that period. This fixed cost might represent the administrative expenses related to managing the backorder, or it might represent a fixed cost related to the loss of goodwill that results from having backorders.

The new objective function, including the delay-dependent backorder costs and the fixed backorder cost is then as follows:

$$\text{Min} \sum_{i=1}^{|I|} \sum_{t=1}^{|T|} \left( sc_{it} y_{it} + boc_{it} u_{it} + \sum_{k=1}^{|T|} c_{ikt} z_{ikt} \right).$$

Note that as we consider both fixed and delay-dependent backorder costs, the traditonal approach, which only considers a unit backlog cost, becomes a specific case of our study (backorder costs and no service level constraint). In practice, it is often difficult to determine the actual cost of a backorder and backlog. This is one of the reasons why from a managerial perspective it sometimes makes more sense to work with service levels. Yet, we will present some computational results in which we model fixed backorder and delay-dependent backorder costs. We will consider variants with only service level constraints, problems with only fixed backorder and delay dependent backorder cost, and problems with both.

## 5. Order management

Different mechanisms can be used to impose inventory flows. The most common ones are LIFO (Last In First Out), FIFO (First In First Out), FEFO (First Expiration First Out) and LEFO (Last Expiration First Out). The last two are relevant when perishability should be taken into account. These mechanisms of inventory management have been studied in Onal et al. [32], where the authors address the economic lot sizing problem with perishable items and consider the four inventory management mechanisms mentioned above. They show that this economic lot sizing problem can be solved in polynomial time for all mechanisms and propose a formulation that can be adapted to every management mechanism.

The inventory management policy rules can also be applied at the order level (instead of at the product inventory level) and hence may affect the service level, especially if this service level is related to the delay in meeting the demand. For this reason, we focus here on a FIFO order management policy which imposes that a demand in period $t$ must be satisfied before satisfying a demand in period $t + 1$. Note that we do not deal with an inventory policy as done traditionally but with an order policy. In Figure 1, we discussed two different interpretations of the solution given. In the first interpretation, demand in period 2 is satisfied from production in period 4, while demand in period 3 is satisfied from production in period 3. In terms of our decision variables, this means $z_{42} = 1$ and $z_{33} = 1$. Such a solution would not be allowed in a FIFO order management system. On the contrary, in the second interpretation, demand in period 2 is satisfied from production in period 3 while demand in period 3 is satisfied from production in period 4. It means that $z_{32} = 1$ and $z_{43} = 1$, a solution which is allowed in a FIFO order management system.

To include this policy in our model, we introduce a binary variable $v_{ikt}$ which takes the value 1 if and only if the production of demand $d_{it}$ starts on or before time period $k$. The FIFO policy is then imposed as follows:

$$v_{ikt} \geq \sum_{j=1}^{k} z_{ijt} \qquad \forall i \in I, k, t \in T \tag{33}$$

$$\sum_{k=1}^{l} z_{ikt} \geq v_{i,l,t+1} \qquad \forall i \in I, l, t \in T. \tag{34}$$

Constraints (33) together with (34) enforce that the $v_{ikt}$ variables take the right values according to their definition. Constraints (34) impose that a demand in period $t$ must be completed before starting production for a later demand. Specifically, if production to satisfy demand for an item $i$ in period $t + 1$ has started on or before period $l$, then the demand of the previous period $t$ must be completely satisfied by production in the periods 1 up to $l$.

## 6. Numerical results

In order to assess the impact of the type of service level constraint, of the intensity of the service level imposed, and of the value of the costs, we conducted computational experiments based on the instances used by Gade and Küçükyavuz [1]. In these instances, holding costs and variable production costs are generated using a uniform distribution on the intervals [1, 5] and [1, 10], respectively. These values are time-variant. As in Gade et al. [1], we let the setup cost be a multiple of the variable production cost. We denote by $\theta$ the ratio between the setup cost and the variable production cost and consider the same values as those used in Gade et al. [1]: $\theta \in \{500, 1000, 2500, 5000\}$. For the service levels, we consider four values for the service level intensities, i.e. parameters alpha, beta and delta used in the PSL, FR and BD service level constraints: $\alpha, \beta, \delta \in \{0.99, 0.95, 0.9, 0.5\}$. For the MD problem, we also consider four values: $\lambda_{max} \in \{1, 2, 3, 8\}$ and for the AD problem we chose $\lambda \in \{0.1, 0.2, 0.3, 0.8\}$. It is important to note that for the first set of experiments, we do not incorporate fixed or delay dependent backorder costs. We consider three items in all our experiments. For each combination of the 10 types of service levels, the four values of the service level intensity, and the four levels of theta, we solve 60 instances with demand coming from the data of Gade and Küçükyavuz [1] for 60 time periods. This leads to a total of 9600 different test instances.

In order to analyze our results, we additionally define two basic cases which will have the same objective function as the tested problems. The first one is the case where backlogging (and hence backordering) is not allowed, called "no backlog" case. The second one is the case where backlogging (and so backordering) is allowed without any service level restriction, which we call the "all backlog" case. For each of the two basic cases, we solve the 60 instances from Gade and Küçükyavuz [1], for all four levels of the ratio $\theta$, resulting in an additional $2\times4\times60 = 480$ instances tested. These two basic cases can be seen as extreme cases which will be helpful in making comparisons with and between the service level constraints. Figure 2 illustrates this by showing at the top an axis representing the solution cost and at the bottom an axis representing the service level intensity. The higher the service level intensity (and so the stricter the service level), the closer one is to the "no backlog" case. Similarly, the lower the service level intensity (and so the looser the service level), the closer one is to the "all backlog" case. Note that for CLSP, the "no backlog" case may lead to infeasible problems. When this happens, we remove the infeasible case from the analysis (it happened twice).


Figure 2: Interpretation of the two base cases in terms of solution cost

We will compare all solutions with respect to four different clusters of indicators:

- CPU time (seconds);
- objective function value;
- structure of the solution;
- key performance indicators (KPIs).

The first two clusters only contain one indicator (CPU time and objective function value). In the third cluster, we include several characteristics linked to the structure of the solution found: the number of items backordered, the total backlog, the number of periods with production for at least one item, the number of periods with positive stock for at least one item and the number of periods with backlog for at least one item. The number of items backordered is computed as $\sum_i \sum_t \sum_{k>t} z_{ikt} d_{it}$, while the total backlog is computed as $\sum_i \sum_t b_{it}$. Note that in the following tables, for the characteristics related to the structure of the solution, we only report the total number of backorders and the total backlog. The reader is referred to Gruson et al. [33] for detailed results. For the KPI cluster, we consider the value of the maximum delay, the global average waiting time, the PSL-G value and the FR-G value. The PSL-G value is computed as $1 - \sum_t u'_{it}/|T|$ while the FR-G value is computed as $1-(\sum_i \sum_t \sum_{k>t} z_{ikt} d_{it} / \sum_i D_i)$. It is important to mention that the maximum delay, the global average waiting time, the PSL-G value and the FR-G value mentioned above are characteristics of a solution that we observe and not service levels that we impose as in (MD-P), (AD-G), (PSL-G) and (FR-G).

We performed our experiments on a 3.07 GHz Intel Xeon processor with only one thread. We used the CPLEX 12.6.1.0 JAVA library. We also turned off CPLEX's parallel mode and used a single thread for the MIP solver: all other CPLEX parameters were set to their default value.

In the following sections, results will be reported with two different tables. At the beginning of each section, a table illustrates the results obtained for the two base cases, for all values of the ratio $\theta$ and on average. Next, we present the results for the set of 9600 test instances. Within each table, we first present the aggregate results for each value of the ratio $\theta$. Next, we present the aggregate results for each value of service level intensity. Finally, we present the aggregate results for each service level type. As MD-P and AD-G constraints do not deal with an intensity expressed as a percentage, the results for the different service level intensities of these two problems are not considered in the service level intensity row of the second set of tables. For all these tables, each column represents a characteristic we observed while each row represents a situation where all results have been aggregated with respect to one fixed parameter. For instance, a row indicating a service level intensity of 99% represents the average evaluation of the characteristics among all instances with an imposed service level of 99%. In the second set of tables, the numbers in parentheses represent the deviation from the average result of the "no backlog" base case for the row where the ratio $\theta$ is fixed, and the deviation from the global average results of the "no backlog" base case otherwise. This is done in order to measure the distance from the best solution in terms of service quality. Thanks to these tables, we will analyze the impact of each component on our solutions and point out the main differences from one service level problem to another. The reader can find detailed results in Gruson et al. [33].

### *6.1. ULSP results*

In the ULSP, no backorder costs are taken into account. In the "all backlog" case without any further service level constraint, there is hence a big incentive to use backlog instead of inventory. Table 2 shows the mean evaluation of the performance indicators of our two base cases over the 60 instances. Detailed results show that for this base case, the number of production periods is not necessarily equal to one. Indeed, it can be better not to use only one of the last periods for the production, but several periods during the planning horizon since the setup and unit production costs are time-variant. The periods with the lowest setup and unit production costs are the most attractive ones. This translates into a really low evaluation of the PSL-G value (3.33%), which means that there is almost always a new backorder at each time period. One can see the large differences in the solution cost between our two base cases. As they are extreme cases, the "all backlog" case is really cheap but leads to poor service characteristics whereas the "no backlog" case is really expensive and leads to the highest possible KPIs.

| | | | | Structure | | KPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| Case | $\theta$ | CPU time | Solution cost | Total back-orders | Total backlog | Maximum delay | Average waiting time | PSL-G value | FR-G value |
| No Backlog | 500 | 0.14 | 249324 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 1000 | 0.15 | 313904 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 2500 | 0.15 | 440517 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 5000 | 0.15 | 586886 | 0 | 0 | 0 | 0 | 100 | 100 |
| | Average | 0.15 | 397658 | 0 | 0 | 0 | 0 | 100 | 100 |
| All Backlog | 500 | 0.27 | 41784 | 26319 | 650525 | 51.67 | 23.28 | 3.33 | 5.8 |
| | 1000 | 0.25 | 46612 | 26205 | 650396 | 51.67 | 23.28 | 3.33 | 6.21 |
| | 2500 | 0.25 | 58924 | 25919 | 649670 | 51.67 | 23.25 | 3.33 | 7.24 |
| | 5000 | 0.26 | 72039 | 25614 | 648919 | 51.67 | 23.22 | 3.33 | 8.33 |
| | Average | 0.26 | 54840 | 26014 | 649878 | 51.67 | 23.26 | 3.33 | 6.9 |

Table 2: Characteristics of the base cases for ULSP

Table 3 shows the results we have obtained for all our different problems. In this table, one can see that the effect of parameter $\theta$ is quite intuitive. Indeed, the higher the setup costs, the higher the objective function value (and the lower the number of production periods). In fact, when production occurs at a certain period $t$, demands for the following periods will be satisfied thanks to inventory and demands of the periods before the next production period after period $t$ are likely to be backordered. Thus, if the ratio $\theta$ increases, there are fewer production periods and hence fewer backorders. The opposite conclusion is also true when the ratio $\theta$ decreases.

In Table 3, one can easily see the gap between a service level of 99% and a service level of 95%: the structure of the solutions we obtain is totally different as can be seen from all characteristics. If we analyze the gap between the other service level intensities, we observe that the solution slowly converges towards a solution that has the same form as the “no backlog” case when we impose stricter service levels. Despite this gap, one can see that the maximum delay remains poor even for an imposed service level of 99%.

The type of service level constraint imposed also has an impact on the form of the solution. One can see the clear distinctions between the families of problems.

For each family, we can point out a main characteristic:

- PSL problems: highest PSL-G value, high waiting time;
- FR problems: high FR-G values, high maximum delay;
- BD problems: highest CPU time, high backorders, low PSL-G value;
- AD-G problem: lowest maximum delay and average waiting time;
- MD-P problem: lowest CPU time, FR-G value and solution cost, highest number of backorders.

For the MD-P, the high number of backorders goes with low average waiting time and maximum delay evaluations, indicating that backorders often occur (PSL-G of 18.84%), but do not last long. Within the FR problems, there are more differences than within the PSL problems: we can clearly distinguish two clusters with FR-I and FR-T on one side and FR-P and FR-G on the other side. All FR measures perform very well on the FR-G criterion, but FR-I and FR-T perform very poorly with respect to the PSL-G criterion, whereas FR-P and FR-G perform much better on this criterion. For the PSL problems, the high average waiting time linked to the relatively high value of FR-G indicates that some orders are filled far from their initial demand period in the time horizon. The solution focuses on backordering in time periods where both demand and costs are high to take full advantage of backorder possibilities. It illustrates the main drawback of this service level.

In Table 3, one can see that when imposing a service level constraint, the performance on the other KPIs is generally poor. This shows that the solution only deals with one service level at a time and this leads to our first main recommendation: it is important to properly define the service level used since they are all different and can lead to completely different solutions. One can also see that, except for the AD-G and MD-P problems, the values of the maximum delay and average waiting time are also very poor. It is also worth mentioning that the number of periods with positive stock and the number of periods with backlog are quite high. This indicates that in our solutions for ULSP, orders are treated regardless of the time period they were placed (no FIFO behavior occurs).

| Case | | CPU time | Solution cost | Structure | | KPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Total back-orders | Total backlog | Maximum delay | Average waiting time | PSL-G value | FR-G value |
| $\theta$ | 500 | 3 | 182114 (-27%) | 7686 | 81058 | 26.82 | 2.9 | 39.53 | 72.49 |
| | 1000 | 3.52 | 229530 (-26.9%) | 7389 | 79534 | 26.42 | 2.85 | 40.78 | 73.55 |
| | 2500 | 3.86 | 325086 (-26.2%) | 6845 | 75760 | 26.37 | 2.71 | 43.36 | 75.5 |
| | 5000 | 4.95 | 438283 (-25.3%) | 6461 | 73244 | 24.68 | 2.62 | 45.4 | 76.88 |
| Service Level Intensity | 99 | 1.37 | 383489 (-3.6%) | 814 | 3882 | 17.79 | 0.15 | 61.11 | 97.09 |
| | 95 | 5.43 | 342343 (-13.9%) | 3388 | 29007 | 32.19 | 1.1 | 50.95 | 87.88 |
| | 90 | 7.54 | 311340 (-21.7%) | 5611 | 53271 | 33.84 | 2.02 | 43.32 | 79.92 |
| | 50 | 4.54 | 167161 (-58%) | 16418 | 256336 | 43.29 | 9.8 | 14.18 | 41.23 |
| Service Level Constraint | PSL-P | 0.4 | 295789 (-25.6%) | 5736 | 109360 | 31.59 | 3.92 | 70.19 | 79.46 |
| | PSL-G | 0.46 | 312877 (-21.3%) | 4896 | 103101 | 31.17 | 3.69 | 83.75 | 82.47 |
| | FR-I | 0.96 | 363102 (-8.7%) | 4166 | 81274 | 42.81 | 2.91 | 4.42 | 85.09 |
| | FR-T | 1.89 | 336640 (-15.3%) | 4399 | 71258 | 38.1 | 2.55 | 4.21 | 84.25 |
| | FR-P | 0.52 | 301931 (-24.1%) | 4610 | 70108 | 36.47 | 3.26 | 65.38 | 83.5 |
| | FR-G | 0.43 | 295836 (-25.6%) | 4610 | 77752 | 38.51 | 3.65 | 66.15 | 83.5 |
| | BD-I | 19.62 | 282105 (-29.1%) | 10569 | 65242 | 10.87 | 2.33 | 27.61 | 62.18 |
| | BD-T | 13.49 | 220387 (-44.6%) | 13477 | 106898 | 24.68 | 3.83 | 17.43 | 51.77 |
| | AD-G | 0.41 | 272009 (-31.6%) | 7005 | 11874 | 3.41 | 0.42 | 46.85 | 74.93 |
| | MD-P | 0.16 | 212940 (-46.5%) | 15250 | 42401 | 4.25 | 1.52 | 18.84 | 45.43 |

Table 3: Characteristics for ULSP with service level constraints

Figure 3 represents on an axis the position of all different problems compared to the base cases, based on the average solution cost among all instances presented in Table 3 for the different problems (row Service Level Constraint), and Table 2 for the base cases. We used a linear interpolation to determine the position of each problem on the axis, considering that the "all backlog" case costs 0 and the "no backlog" case costs 1. One can directly see that FR-I seems to be the closest problem to the no backlog case even if its PSL-G value is really low. From a managerial perspective, Figure 3 shows the price one has to pay and the distance from the "no backlog" case (ideal solution). Note that one should not conclude directly that it is best to impose a FR-I constraints since in Figure 3 the analysis is only based on the objective function value and not on all characteristics presented in Table 2.

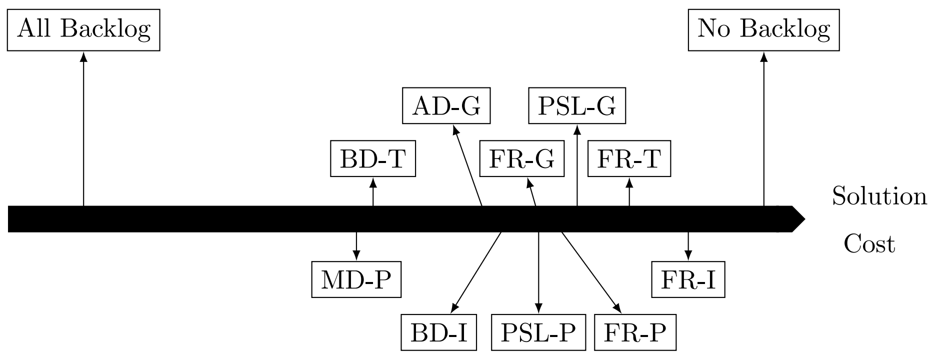


Figure 3: Average position of the different service level problems in terms of solution cost

*6.2. ULSP results with stationary costs*

We ran the same computational experiments with setup, production and holding costs that are not time dependent. They vary from one instance to the other but are constant over time within one instance. Their values are set as the average costs over the complete planning horizon in the initial data used. Table 4 shows the characteristics for our two base cases with stationary costs. Contrary to ULSP with non stationary costs, in ULSP with stationary costs, the "all backlog" case has only one production period. This particular period is the last one, to avoid paying for holding costs. This is possible because we do not have capacity constraints. Still for this base case, there is no impact of the value of $\theta$ since it is always profitable to have the same production planning, regardless of the costs. For both our base cases, there is an increase in the solution cost compared to the case with non-stationary costs since one can no longer take advantage of the cost differences over time (more visible for the "all backlog" case). Note that the CPU time decreases compared to ULSP with non stationary costs.

| Case | θ | CPU time | Solution cost | Structure: Total back-orders | Structure: Total backlog | KPI: Maximum delay | KPI: Average waiting time | KPI: PSL-G value | KPI: FR-G value |
|---|---|---|---|---|---|---|---|---|---|
| No Backlog | 500 | 0.08 | 375896 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 1000 | 0.08 | 487973 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 2500 | 0.08 | 715323 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 5000 | 0.09 | 977355 | 0 | 0 | 0 | 0 | 100 | 100 |
| | Average | 0.08 | 639137 | 0 | 0 | 0 | 0 | 100 | 100 |
| All Backlog | 500 | 0.16 | 161835 | 27449 | 822845 | 59 | 29.45 | 1.67 | 1.76 |
| | 1000 | 0.16 | 170081 | 27449 | 822845 | 59 | 29.45 | 1.67 | 1.76 |
| | 2500 | 0.16 | 194819 | 27449 | 822845 | 59 | 29.45 | 1.67 | 1.76 |
| | 5000 | 0.17 | 236049 | 27449 | 822845 | 59 | 29.45 | 1.67 | 1.76 |
| | Average | 0.1625 | 190696 | 27449 | 822845 | 59 | 29.45 | 1.67 | 1.76 |

Table 4: Characteristics of the base cases for ULSP with stationary costs

Table 5 shows the average results for the ULSP with stationary costs. In Table 5, one can notice that the gap we had in Table 3 between service level intensities of 99% and 95% is now smaller but still visible. Compared to the ULSP with non stationary costs, one can easily observe the large increase in the CPU time (up to 62 times more for FR-T problem) and in the solution cost (up to 2.28 times more for a service level intensity of 50%). This is because the solution can no longer take advantage of cost fluctuations during the planning horizon. The stationary costs also have an impact on the other characteristics, but the impact depends on the service level constraint imposed. For instance, the number of backorders decreases for the FR-I problem whereas it increases for the PSL-G problem. Thus, stationary costs tend to highlight differences from one service level problem to another.

| Case | | CPU time | Solution cost | Structure | | KPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Total back-orders | Total backlog | Maximum delay | Average waiting time | PSL-G value | FR-G value |
| $\theta$ | 500 | 67.97 | 316767 (-15.7%) | 7960 | 73356 | 28.11 | 2.63 | 35.11 | 71.51 |
| | 1000 | 73.08 | 406873 (-16.6%) | 7491 | 71951 | 27.01 | 2.58 | 38.04 | 73.19 |
| | 2500 | 76.88 | 596236 (-16.6%) | 6710 | 71914 | 26.02 | 2.57 | 43.74 | 75.99 |
| | 5000 | 76.89 | 820473 (-16.1%) | 6082 | 71011 | 24.71 | 2.54 | 48.31 | 78.24 |
| Service Level Intensity | 99 | 65 | 627681 (-1.8%) | 756 | 4511 | 19.12 | 0.16 | 61.62 | 97.29 |
| | 95 | 82.73 | 591680 (-7.4%) | 3274 | 23535 | 29.69 | 0.84 | 48.93 | 88.28 |
| | 90 | 91.35 | 559473 (-12.5%) | 5487 | 45436 | 33.43 | 1.63 | 40.68 | 80.36 |
| | 50 | 129.06 | 381756 (-40.3%) | 16566 | 263983 | 46.9 | 9.45 | 13.34 | 40.71 |
| Service Level Constraint | PSL-P | 1.88 | 543817 (-14.9%) | 5987 | 95959 | 25.01 | 3.44 | 68.85 | 78.56 |
| | PSL-G | 2.64 | 555175 (-13.1%) | 5256 | 83436 | 25.73 | 2.98 | 83.75 | 81.18 |
| | FR-I | 33.25 | 603361 (-5.6%) | 3797 | 68592 | 47.24 | 2.45 | 8.35 | 86.41 |
| | FR-T | 117.27 | 572061 (-10.5%) | 4359 | 60484 | 39.25 | 2.17 | 6.78 | 84.4 |
| | FR-P | 1.56 | 547476 (-14.3%) | 4610 | 85010 | 34.54 | 3.05 | 58.12 | 83.5 |
| | FR-G | 0.6 | 546011 (-14.6%) | 4610 | 85075 | 34.37 | 3.05 | 58.97 | 83.5 |
| | BD-I | 279.05 | 507175 (-20.6%) | 10014 | 75402 | 11.33 | 2.7 | 30.1 | 64.17 |
| | BD-T | 300.05 | 446103 (-30.2%) | 13535 | 120972 | 40.85 | 4.33 | 14.22 | 51.57 |
| | AD-G | 0.6 | 526555 (-17.6%) | 7060 | 11874 | 3.27 | 0.42 | 41.06 | 74.74 |
| | MD-P | 0.18 | 455539 (-28.7%) | 14874 | 44352 | 4.25 | 1.59 | 26.45 | 46.77 |

Table 5: Characteristics for ULSP with stationary costs

In general, we can give the same remarks and conclusions as for ULSP with time-varying costs in general. The main differences are in the evaluation of some characteristics. We see more backorders than previously because there are no differences in the costs: one just takes into account the demand to choose the periods where backlogs should occur. This helps focus on time periods with high demand and thus it leads to more backorders.

Figure 4 represents the position of all problems among all instances in terms of average solution cost. One can see that, except for MD-P and BD-T problems, the order is the same as in Figure 3 for ULSP results with dynamic costs. We note here that the stationary costs make the problems closer to the no backlog case, still in terms of solution cost.

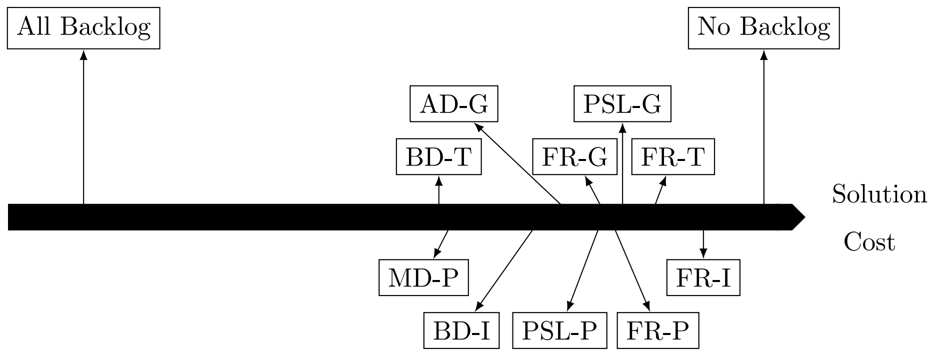


Figure 4: Average position of the different service level problems in terms of solution cost

*6.3. Backorder costs*

For testing purposes, we chose to relate the backorder costs to the other types of costs that are present in the model. Thus, we set fixed backorder costs $boc_{it}$ to a portion of the setup costs and we set delay dependent backorder costs $bdc_{it}$ to a portion of the holding costs. Note that these two sets of costs are time variant. For both costs, we tested a wide range of values reflecting various potential situations. We set fixed backorder costs as follows: $boc_{it} = \nu \times sc_{it}$ with $\nu \in \{0.05, 0.1, 0.2, 0.5\}$. We set the variable delay-dependent backorder costs as follows: $bdc_{i,k-t} = \mu f(k-t) \times hc_{i,t}$, where $k > t$, $\mu \in \{0.5, 1.0, 2.0\}$ and $f$ is one of the following functions: $f : x \rightarrow x$, $f : x \rightarrow x^{\frac{5}{4}}$, $f : x \rightarrow x^{\frac{3}{2}}$ and $f : x \rightarrow x^2$. We chose convex functions to take into account the assumption that a two-period wait is worse than two backorders of one period each. This idea of having non-linear shortage costs has also been used in Badia and Sangüesa [34] in the context of inventory models with stochastic lead times.

We thus ran the same computational experiments with fixed backorder costs and delay dependent backorder costs (and no service level constraints). This leads to drastic changes in the form of solutions, with a clear improvement in the KPIs. We globally observe the following results, based on the detailed results:

- the solutions do not often take advantage of the backlogging and backordering flexibility because it may cost more than producing on time;

- increase in the number of production periods;
- decrease in the number of backlogging periods;
- fewer backorders;
- lower waiting time;
- shorter maximum delay;
- better PSL-G and FR-G values.

As a result, the solutions provide better service levels but the objective function value increases. There is also a better balance between the various KPIs. This happens for two different reasons. If backorder costs are low, backorders occur and the increase in the cost is the consequence of the new backorder costs. On the contrary, if backorder costs are high, backorders do not occur and the increase is the consequence of the basic costs (production, setup and inventory holding costs).

Table 6 shows the characteristics of the "all backlog" base case. We obviously obtained the same results for the no backlog base case as in Table 2 and thus we do not display them. Note that with the backorder costs, the "all backlog" base case can be seen as an alternative to having service level constraints. Indeed, the backorder costs act as a surrogate to the service level. In Table 6, we display the average results when the cost parameters change: $\theta$, $\mu$, $\nu$, and the delay dependent cost function.

The last row of Table 6 illustrates the results we obtained by solving our instances in a traditional way, i.e. with formulation (12)-(16), using traditional unit backlog costs. We used a linear delay-dependant cost function and imposed $\mu = 1$ and $\nu = 0$, so that there are no fixed backorder costs. We thus only have a cost linked to the backlog for each item and each time period as done in Pochet and Wolsey [2]. We note that, in this traditional case, the proportion of backorder costs is really high compared to the ones we obtain for the "all backlog" case with backorder costs in Table 6, mostly because of the absence of fixed cost. The solution is also more expensive than for the "all backlog" case for the same reason. This shows that, in the traditional way, one can still take full advantage of backorder possibilities, and this reflects in the KPIs whose evaluations are poor.

| Case | | CPU time | Solution cost | Total back-order costs (%) | Structure: Total back-orders | Structure: Total backlog | KPI: Maximum delay | KPI: Average waiting time | KPI: PSL-G value | KPI: FR-G value |
|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ | 500 | 0.29 | 238036 | 10.37 | 3246 | 5852 | 1.76 | 0.21 | 78.92 | 88.38 |
| | 1000 | 0.29 | 288266 | 14.21 | 4613 | 9724 | 2.53 | 0.35 | 71.52 | 83.49 |
| | 2500 | 0.29 | 376903 | 18.9 | 6402 | 16967 | 4.02 | 0.61 | 61.39 | 77.09 |
| | 5000 | 0.3 | 471824 | 22.31 | 7701 | 25518 | 5.76 | 0.91 | 54.99 | 72.44 |
| Delay dependant cost function | $x$ | 0.24 | 333495 | 20.61 | 6763 | 21435 | 4.66 | 0.77 | 61.08 | 75.79 |
| | $x^{\frac{5}{4}}$ | 0.19 | 339615 | 18.23 | 5993 | 17309 | 3.99 | 0.62 | 64.52 | 78.55 |
| | $x^{\frac{3}{2}}$ | 0.23 | 346641 | 15.28 | 5104 | 11981 | 3.16 | 0.43 | 68.31 | 81.73 |
| | $x^2$ | 0.24 | 355278 | 11.68 | 4101 | 7337 | 2.26 | 0.26 | 72.91 | 85.32 |
| Proportion of backordering costs | $\mu = 2, \nu = 0.5$ | 0,29 | 387998 | 2,76 | 519 | 784 | 0,69 | 0,03 | 96,1 | 98,14 |
| | $\mu = 2, \nu = 0.2$ | 0,28 | 367862 | 10 | 2677 | 5486 | 2,37 | 0,2 | 81,9 | 90,42 |
| | $\mu = 2, \nu = 0.1$ | 0,3 | 341935 | 18,11 | 5721 | 13079 | 3,65 | 0,47 | 63,42 | 79,53 |
| | $\mu = 2, \nu = 0.05$ | 0,27 | 314891 | 23,26 | 8508 | 21128 | 4,52 | 0,76 | 47,39 | 69,55 |
| | $\mu = 1, \nu = 0.5$ | 0,3 | 387244 | 3,08 | 633 | 1018 | 0,84 | 0,04 | 95,33 | 97,74 |
| | $\mu = 1, \nu = 0.2$ | 0,29 | 362623 | 12,2 | 3377 | 7651 | 2,93 | 0,27 | 77,92 | 87,91 |
| | $\mu = 1, \nu = 0.1$ | 0,3 | 329664 | 22,17 | 7243 | 18078 | 4,51 | 0,65 | 55,65 | 74,08 |
| | $\mu = 1, \nu = 0.05$ | 0,27 | 295496 | 28,55 | 10798 | 29329 | 5,61 | 1,05 | 37,02 | 61,36 |
| | $\mu = 0.5, \nu = 0.5$ | 0,32 | 386683 | 3,27 | 713 | 1227 | 0,95 | 0,04 | 94,78 | 97,45 |
| | $\mu = 0.5, \nu = 0.5$ | 0,29 | 357568 | 14,25 | 4069 | 10240 | 3,58 | 0,37 | 73,7 | 85,44 |
| | $\mu = 0.5, \nu = 0.5$ | 0,32 | 317399 | 26,39 | 8809 | 25245 | 5,59 | 0,9 | 48,13 | 68,48 |
| | $\mu = 0.5, \nu = 0.5$ | 0,28 | 275724 | 33,36 | 12820 | 40921 | 6,97 | 1,46 | 29,12 | 54,12 |
| Traditional | $\mu = 1, \nu = 0$ | 0.27 | 202041 | 33.38 | 17394 | 67968 | 9.37 | 2.44 | 11.7 | 37.74 |

Table 6: Characteristics for all backlog base case with backorder costs

Next, we solved the problem in which both a service level and backordering costs are present. We present the results for the PSL-P problem with the additional backorder costs. Table 7 provides the results for all values of the ratio $\theta$, all service level intensities, all delay-dependent cost functions and all combinations of ratios $\mu$ and $\nu$. Note that as we obtained similar results for the other service level constraints imposed, we just illustrate one problem (PSL-P). It is worth mentionning that in the traditional case where we only consider a unit backlog cost, the solution we get is clearly cheaper but has really poor KPIs compared to the solutions we get with our backorder costs. Following a detailed analysis for other service levels, we note that the FR-I problem is much more affected by the backorder costs than the other problems. Indeed, almost no backorders occur for this problem (only a few instances still have backorders).

| Case | | CPU time | Solution cost | Total back-order costs (%) | Structure: Total back-orders | Structure: Total backlog | KPI: Maximum delay | KPI: Average waiting time | KPI: PSL-G value | KPI: FR-G value |
|---|---|---|---|---|---|---|---|---|---|---|
| $\theta$ | 500 | 0.21 | 242819 (-2.6%) | 4.96 | 1540 | 2531 | 1.08 | 0.09 | 89.54 | 94.49 |
| | 1000 | 0.21 | 299331 (-4.6%) | 6.3 | 2066 | 3832 | 1.48 | 0.14 | 86.81 | 92.61 |
| | 2500 | 0.22 | 404997 (-8.1%) | 7.68 | 2723 | 6029 | 2.07 | 0.22 | 82.81 | 90.25 |
| | 5000 | 0.28 | 524858 (-10.6%) | 8.25 | 3127 | 8327 | 2.66 | 0.3 | 80.99 | 88.81 |
| Service Level Intensity | 99 | 0.13 | 397658 (-0%) | 0 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 95 | 0.25 | 369859 (-7%) | 4 | 1413 | 2121 | 1.66 | 0.08 | 90.6 | 94.94 |
| | 90 | 0.3 | 360687 (-9.3%) | 6.86 | 2580 | 4338 | 2.15 | 0.16 | 82.72 | 90.76 |
| | 50 | 0.24 | 343800 (-13.5%) | 16.32 | 5463 | 14259 | 3.49 | 0.51 | 66.83 | 80.45 |
| Delay dependant cost function | $x$ | 0.3 | 364744 (-8.3%) | 7.86 | 2718 | 7076 | 2.24 | 0.25 | 83.45 | 90.27 |
| | $x^{\frac{5}{4}}$ | 0.16 | 366749 (-7.8%) | 7.24 | 2496 | 5921 | 1.99 | 0.21 | 84.47 | 91.06 |
| | $x^{\frac{3}{2}}$ | 0.23 | 368866 (-7.2%) | 6.52 | 2263 | 4536 | 1.7 | 0.16 | 85.47 | 91.9 |
| | $x^2$ | 0.21 | 371645 (-6.5%) | 5.56 | 1978 | 3185 | 1.36 | 0.11 | 86.75 | 92.92 |
| Proportion of backordering costs | $\mu = 2, \nu = 0.5$ | 0.21 | 390459 (-1.8%) | 1.99 | 369 | 551 | 0.51 | 0.02 | 97.21 | 98.68 |
| | $\mu = 2, \nu = 0.2$ | 0.24 | 377747 (-5%) | 5.63 | 1519 | 2670 | 1.45 | 0.1 | 89.69 | 94.56 |
| | $\mu = 2, \nu = 0.1$ | 0.24 | 364712 (-8.3%) | 8.16 | 2705 | 5168 | 1.98 | 0.19 | 82.27 | 90.32 |
| | $\mu = 2, \nu = 0.05$ | 0.23 | 353196 (-11.2%) | 8.63 | 3490 | 7186 | 2.22 | 0.26 | 77.65 | 87.5 |
| | $\mu = 1, \nu = 0.5$ | 0.19 | 389925 (-1.9%) | 2.19 | 444 | 705 | 0.61 | 0.03 | 96.71 | 98.41 |
| | $\mu = 1, \nu = 0.2$ | 0.2 | 375175 (-5.7%) | 6.35 | 1802 | 3457 | 1.7 | 0.12 | 88.05 | 93.55 |
| | $\mu = 1, \nu = 0.1$ | 0.26 | 359841 (-9.5%) | 9.14 | 3182 | 6715 | 2.33 | 0.24 | 79.9 | 88.61 |
| | $\mu = 1, \nu = 0.05$ | 0.24 | 346423 (-12.9%) | 9.63 | 4100 | 9469 | 2.61 | 0.34 | 75.05 | 85.32 |
| | $\mu = 0.5, \nu = 0.5$ | 0.2 | 389540 (-2%) | 2.31 | 493 | 829 | 0.69 | 0.03 | 96.38 | 98.24 |
| | $\mu = 0.5, \nu = 0.2$ | 0.23 | 373064 (-6.2%) | 6.95 | 2053 | 4333 | 1.98 | 0.16 | 86.49 | 92.65 |
| | $\mu = 0.5, \nu = 0.1$ | 0.27 | 355603 (-10.6%) | 10.08 | 3613 | 8756 | 2.73 | 0.31 | 77.88 | 87.07 |
| | $\mu = 0.5, \nu = 0.05$ | 0.25 | 340326 (-14.4%) | 10.48 | 4597 | 12316 | 3.07 | 0.44 | 73.17 | 83.54 |
| Traditional | $\mu = 1, \nu = 0$ | 0.37 | 318548 (-19.9%) | 5.04 | 5284 | 16054 | 3.51 | 0.57 | 70.67 | 81.08 |

Table 7: Characteristics for PSL-P with backorder costs

One can see in Tables 6 and 7 that the CPU time is approximately similar for each combination of parameters. We can also see that the impact of the delay dependent cost function is rather limited in terms of improvements of the KPIs: the fixed costs have much more impact on the form of the solution and on the decision to backorder or not. Also, by analyzing Tables 6 and 7, we can draw some conclusions about backorder costs. First, the higher the ratio $\theta$, the higher the total proportion of backorder cost. Indeed, if it is expensive to produce, one will produce less and thus backorder more. Then, the delay dependent cost function acts as one could expect: the more convex the function, the lower the proportion of backorder costs since it costs too much to backorder. Finally, the higher the service level intensity, the lower the total proportion of backorder cost since one will have many more backorders.

Considering that the "all backlog" base case with backorder costs represents an alternative for a service level constraint, one can see that it does not provide an entirely good solution in terms of all KPIs. This shows the importance of the service level constraints, since we took a wide range of possibilities for backordering costs. As a matter of fact, for the problems with service level constraints, having backorder costs tends to stabilize solutions in terms of the various KPIs even if we do not really know the values of these costs. This means that, instead of putting some constraints, one could put an artificial cost and this can provide solutions that are acceptable for all KPIs. This is actually one of the conclusions we can draw by comparing ULSP results, ULSP results with backorder costs and service level constraint, and ULSP results with backorder costs. Service level constraints alone deal with one specific feature of the solution while backorder costs alone give a more balanced solution in terms of all KPIs. When we add both service level constraints and backorder costs, we still get a balanced solution but with a better value of the feature targetted by the service level constraint.

### *6.4. FIFO order management*

We ran the same experiments including the order management constraint (34) and with the time variant costs. Table 8 shows the evaluation of the characteristics for the base cases. One can see that the results are exactly the same as the ones we obtained for ULSP experiments with non stationary costs (for the "all backlog" base case, it is because all production takes place at the end of the time horizon anyway since, in terms of solution cost, it is better to backorder). The only difference comes from the CPU time, which is higher when we add a FIFO order management policy.

| Case | $\theta$ | CPU time | Solution cost | Structure | | KPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Total back-orders | Total backlog | Maximum delay | Average waiting time | PSL-G value | FR-G value |
| No Backlog | 500 | 0.83 | 249324 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 1000 | 0.8 | 313904 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 2500 | 0.81 | 440517 | 0 | 0 | 0 | 0 | 100 | 100 |
| | 5000 | 0.85 | 586886 | 0 | 0 | 0 | 0 | 100 | 100 |
| | Average | 0.82 | 397658 | 0 | 0 | 0 | 0 | 100 | 100 |
| All Backlog | 500 | 1.62 | 41784 | 26319 | 650525 | 51.67 | 23.28 | 3.33 | 5.8 |
| | 1000 | 1.59 | 46612 | 26205 | 650396 | 51.67 | 23.28 | 3.33 | 6.21 |
| | 2500 | 1.59 | 58924 | 25919 | 649670 | 51.67 | 23.25 | 3.33 | 7.24 |
| | 5000 | 1.69 | 72039 | 25614 | 648919 | 51.67 | 23.22 | 3.33 | 8.33 |
| | Average | 1.62 | 54840 | 26014 | 649878 | 51.67 | 23.26 | 3.33 | 6.9 |

Table 8: Characteristics of the base cases for ULSP with FIFO policy

Table 9 shows the average evaluations for ULSP with FIFO order management and service levels. One can see that the FIFO order management helps to obtain better results. There is an important decrease in the maximum waiting time and average delay compared to the results of ULSP. For instance, the maximum delay is decreased by 42.81 times for the FR-I problem and reaches 1 time period. In the same vein, still for the FR-I problem, the average waiting time goes from 2.91 for ULSP to 0.03 for ULSP with FIFO order management. Indeed, the FIFO order management reduces the length of backorders. Detailed results show that it also translates into a decrease in the number of periods with backlog. Indeed, in ULSP, demands that were filled far from their initial time period are now filled much faster. The difference of backlog duration between the solution in ULSP and the solution in ULSP with FIFO order management is thus equivalent to a decrease of backlogging periods.

The FIFO order leads to an increase of CPU time. The solution cost also increases, but never more than 12% (service level intensity of 50%). For all the other characteristics, the impact of the FIFO order management depends on the problem and thus on the service constraint imposed.

By comparing Tables 3 and 9, one can see that the FIFO order management policy does not much affect the results for MD-P and AD-G problems since these two service levels already deal with the time dimension. Furthermore, the KPIs related to BD are not much affected by the introduction of FIFO (except maximum delay for the BD-T).

| Case | | CPU time | Solution cost | Structure | | KPI | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | Total back-orders | Total backlog | Maximum delay | Average waiting time | PSL-G value | FR-G value |
| $\theta$ | 500 | 16.83 | 186874 (-25%) | 7254 | 32342 | 5.24 | 1.16 | 45.9 | 74.04 |
| | 1000 | 18.64 | 235098 (-25.1%) | 6882 | 32093 | 5.33 | 1.15 | 48.8 | 75.37 |
| | 2500 | 22.07 | 332117 (-24.6%) | 6259 | 32170 | 5.6 | 1.15 | 53.08 | 77.6 |
| | 5000 | 26.9 | 446723 (-23.9%) | 5829 | 33328 | 5.91 | 1.19 | 56.05 | 79.14 |
| Service Level Intensity | 99 | 4.9 | 384187 (-3.4%) | 755 | 886 | 1.12 | 0.03 | 73.57 | 97.3 |
| | 95 | 19.83 | 346292 (-12.9%) | 3121 | 6158 | 2.88 | 0.22 | 63.19 | 88.83 |
| | 90 | 36.73 | 318724 (-19.8%) | 5106 | 14296 | 4.44 | 0.51 | 55.37 | 81.73 |
| | 50 | 41.94 | 187379 (-52.9%) | 14481 | 118539 | 15.89 | 4.25 | 20.89 | 48.17 |
| Service Level Constraint | PSL-P | 5.1 | 302809 (-23.9%) | 5201 | 23016 | 4.61 | 0.82 | 69.98 | 81.38 |
| | PSL-G | 6.19 | 319601 (-19.6%) | 4363 | 20663 | 4.48 | 0.74 | 83.75 | 84.38 |
| | FR-I | 4.43 | 388962 (-2.2%) | 892 | 892 | 1.00 | 0.03 | 55.31 | 96.81 |
| | FR-T | 19.29 | 356234 (-10.4%) | 3076 | 10297 | 3.49 | 0.37 | 41.88 | 88.99 |
| | FR-P | 6.37 | 304183 (-23.5%) | 4610 | 22634 | 5.67 | 0.81 | 65.14 | 83.5 |
| | FR-G | 4.82 | 297595 (-25.2%) | 4610 | 38943 | 6.97 | 1.4 | 65.38 | 83.5 |
| | BD-I | 92.37 | 282109 (-29.1%) | 10751 | 65264 | 9.79 | 2.34 | 27.26 | 61.53 |
| | BD-T | 68.21 | 221671 (-44.3%) | 13423 | 98047 | 12.67 | 3.51 | 17.36 | 51.97 |
| | AD-G | 3.12 | 272010 (-31.6%) | 7002 | 11874 | 3.41 | 0.42 | 46.86 | 74.94 |
| | MD-P | 1.23 | 212940 (-46.5%) | 15485 | 43987 | 4.25 | 1.57 | 18.79 | 44.58 |

Table 9: Characteristics for ULSP with FIFO policy and service levels

Figure 5 represents the position of all problems among all instances in terms of average solution cost. One can see that the order is the same as in Figure 4 for ULSP results with stationary costs. We note here that, as for the stationary costs, the FIFO order management policy makes the problems closer to the no backlog case in terms of solution cost. In Figure 5, the closeness between the FR-I problem and the no backlog case is clearly visible as depicted in the characteristics of the FR-I problem in Table 9.

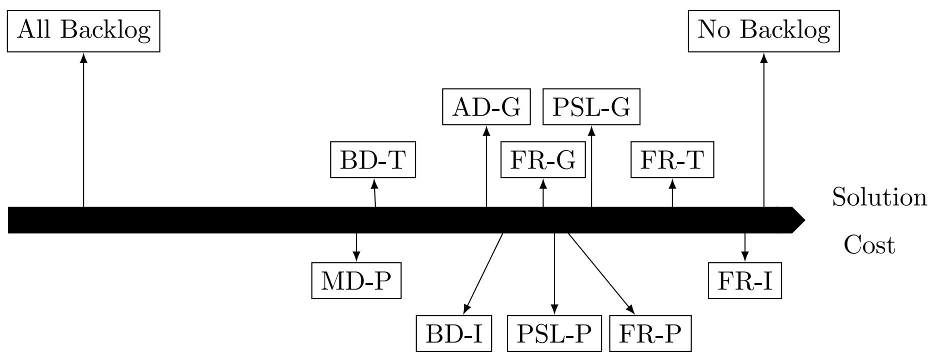


Figure 5: Average position of the different service level problems in terms of solution cost

### *6.5. CLSP results*

We have finally run experiments with capactiy constraints added to (12)-(16). For a review of models and algorithms used to solve the CLSP we refer the reader to Karimi et al. [35]. In our case, we decided to set the production capacity as a given factor of the average demand over all items and time periods. If we denote by $C$ the capacity factor, the capacity constraints are as follows:

$$\sum_i \sum_t z_{ikt} d_{it} \leq C \frac{\sum_i D_i}{|T|} \quad \forall k \in T. \tag{35}$$

We added to the CLSP the service level constraints and ran the same experiments as in Section 6.1. For the experiments, we considered three different values for the capacity factor $C$: $C \in \{2, 1.75, 1.5\}$. These values ensure that we obtained feasible instances, except for one instance of the "no backlog" case. We imposed a time limit of 10 minutes for each experiment. Note that this time limit is reached for more than 90% of the instances.

Table 10 shows the evaluation of the characteristics for the two base cases and Table 11 shows the average evaluations for CLSP. In these two tables, the column GAP indicates the average optimality gap obtained. We can directly see the consequences of the capacity constraints on the results:

- increase in the objective function value since there is less flexibility than in the ULSP to choose the production periods;

- decrease in the average waiting time compared to the one we observed in ULSP results (generally, and also for the maximum waiting time).

A deeper analysis shows that the number of production periods will be close to $|T|/C$, which is the minimum possible to satisfy all demand. Nevertheless, it is impossible to identify a clear impact of the capacity constraint on the KPIs. Indeed, we have improvements or deteriorations depending on the service level considered and on the KPI we look at.

Note that with the capacity constraints, one could be forced to backorder a portion of the demand. Indeed, consider a case where $d_t$ equals the production capacity for all $t < t'$ and $d_{t'}$ is strictly higher than the production capacity. Then, we must backorder the portion of the demand that exceeds the production capacity.

| | | | | | Structure | | KPI | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Case | Capacity | $\theta$ | CPU time | Solution cost | Total back-orders | Total backlog | Maximum delay | Average waiting time | PSL-G value | FR-G value | GAP |
| No Backlog | 2 | 500 | 0.67 | 262417 | 0 | 0 | 0 | 0 | 100 | 100 | 0.002 |
| | | 1000 | 3.6 | 340726 | 0 | 0 | 0 | 0 | 100 | 100 | 0.01 |
| | | 2500 | 54.73 | 524329 | 0 | 0 | | 0 | 100 | 100 | 0.01 |
| | | 5000 | 350.58 | 785848 | 0 | 0 | 0 | 0 | 100 | 100 | 0.43 |
| | 1.75 | 500 | 2.15 | 270818 | 0 | 0 | 0 | 0 | 100 | 100 | 0.01 |
| | | 1000 | 16.81 | 355796 | 0 | 0 | 0 | 0 | 100 | 100 | 0.01 |
| | | 2500 | 255.68 | 562923 | 0 | 0 | 0 | 0 | 100 | 100 | 0.18 |
| | | 5000 | 544.32 | 863784 | 0 | 0 | 0 | 0 | 100 | 100 | 1.34 |
| | 1.5 | 500 | 11.56 | 274865 | 0 | 0 | 0 | 0 | 100 | 100 | 0.01 |
| | | 1000 | 90.65 | 367553 | 0 | 0 | 0 | 0 | 100 | 100 | 0.01 |
| | | 2500 | 525.7 | 602152 | 0 | 0 | 0 | 0 | 100 | 100 | 1.1 |
| | | 5000 | 575.53 | 952893 | 0 | 0 | 0 | 0 | 100 | 100 | 2.36 |
| | Average | | 202.67 | 513675 | 0 | 0 | 0 | 0 | 100 | 100 | 0.46 |
| All Backlog | 2 | 500 | 2.36 | 225405 | 2635 | 2793.91 | 1.57 | 0.11 | 9.46 | 9.43 | 0.01 |
| | | 1000 | 11.9 | 287907 | 2560 | 2793.91 | 1.76 | 0.11 | 9.17 | 9.16 | 0.03 |
| | | 2500 | 140.15 | 429857 | 2311 | 2793.91 | 2.09 | 0.11 | 8.29 | 8.27 | 0.95 |
| | | 5000 | 475.88 | 626205 | 2112 | 2793.91 | 2.34 | 0.12 | 7.58 | 7.56 | 1.71 |
| | 1.75 | 500 | 6.92 | 231682 | 2656 | 2794 | 1.59 | 0.12 | 35.5 | 9.51 | 0.01 |
| | | 1000 | 54.55 | 299013 | 2539 | 2794 | 1.84 | 0.12 | 35.78 | 9.09 | 0.3 |
| | | 2500 | 449.32 | 459293 | 2281 | 2794 | 2.15 | 0.11 | 33.94 | 8.16 | 1.78 |
| | | 5000 | 593.92 | 687051 | 2136 | 2794 | 2.28 | 0.12 | 30.25 | 7.64 | 2.3 |
| | 1.5 | 500 | 40.98 | 242258 | 2652 | 2794 | 1.68 | 0.12 | 38.56 | 9.49 | 0.09 |
| | | 1000 | 314.69 | 318245 | 2523 | 2794 | 1.91 | 0.12 | 39 | 9.03 | 0.79 |
| | | 2500 | 594.11 | 505712 | 2325 | 2794 | 2.17 | 0.12 | 35.78 | 8.32 | 1.94 |
| | | 5000 | 599.27 | 784195 | 2169 | 2794 | 42.34 | 0.13 | 32.67 | 7.76 | 2.52 |
| | Average | | 273.67 | 424735 | 2408 | 2794 | 1.98 | 0.12 | 65.8 | 91.38 | 1.04 |

Table 10: Characteristics of the base cases for CLSP

| Case | | CPU time | Solution cost | Structure | | KPI | | | | GAP |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | Total back-orders | Total backlog | Maximum delay | Average waiting time | PSL-G value | FR-G value | |
| $\theta$ | 500 | 123.71 | 193980 (-28%) | 9188 | 53352 | 21.88 | 1.67 | 35.78 | 67.11 | 0.11 |
| | 1000 | 283.94 | 256021 (-27.8%) | 8776 | 52290 | 23.01 | 1.7 | 36.87 | 68.59 | 0.64 |
| | 2500 | 477.78 | 412350 (-26.8%) | 7967 | 49326 | 23.55 | 1.71 | 39.36 | 71.49 | 2.34 |
| | 5000 | 572.06 | 645886 (-25.5%) | 7242 | 46759 | 23.43 | 1.62 | 42.06 | 74.08 | 1.79 |
| Service Level Intensity | 99 | 335.5 | 461082 (-10.2%) | 3315 | 20981 | 20.95 | 0.69 | 59.89 | 88.14 | 2.22 |
| | 95 | 374.51 | 407969 (-20.6%) | 5704 | 42345 | 31.19 | 1.5 | 48.31 | 79.59 | 2.19 |
| | 90 | 375.27 | 375050 (-27%) | 7694 | 58771 | 31.15 | 2.04 | 40.66 | 72.46 | 1.92 |
| | 50 | 441.21 | 278095 (-45.9%) | 15884 | 109841 | 28.06 | 3.4 | 12.15 | 43.15 | 1.10 |
| Service Level Constraint | PSL-P | 330.01 | 391085 (-23.9%) | 5439 | 55612 | 25.81 | 1.77 | 68.75 | 80.52 | 1.89 |
| | PSL-G | 366.12 | 412437 (-19.7%) | 4494 | 58452 | 29.21 | 2.14 | 83.75 | 83.91 | 2.48 |
| | FR-I | 339.28 | 464822 (-9.5%) | 3600 | 33690 | 37.75 | 1.26 | 4.25 | 87.12 | 1.20 |
| | FR-T | 403.21 | 243633 (-52.6%) | 20591 | 129233 | 27.65 | 3.94 | 2.99 | 26.3 | 1.01 |
| | FR-P | 295.64 | 392852 (-23.5%) | 4605 | 48827 | 33.38 | 1.53 | 60.35 | 83.52 | 1.13 |
| | FR-G | 285.77 | 387727 (-24.5%) | 4610 | 51191 | 32.62 | 1.6 | 61.98 | 83.5 | 1.03 |
| | BD-I | 544.03 | 414186 (-19.4%) | 9370 | 31981 | 8.56 | 1.18 | 24.12 | 66.47 | 4.33 |
| | BD-T | 488.92 | 337651 (-34.3%) | 12484 | 54892 | 27.72 | 1.83 | 15.83 | 55.33 | 2.95 |
| | AD-G | 313.4 | 359352 (-30%) | 6924 | 11873 | 3.94 | 0.43 | 33.2 | 75.22 | 1.1 |
| | MD-P | 296.46 | 321762 (-37.4%) | 14558 | 38243 | 4.25 | 1.36 | 11.42 | 47.9 | 0.96 |
| Capacity | 2 | 268.47 | 340491 (-33.7%) | 8449 | 53571 | 22.88 | 1.79 | 39.02 | 69.76 | 2.48 |
| | 1.75 | 362.53 | 371626 (-27.7%) | 8312 | 50826 | 22.99 | 1.68 | 38.47 | 70.25 | 1.76 |
| | 1.5 | 462.11 | 419061 (-18.4%) | 8120 | 46898 | 23.02 | 1.55 | 38.07 | 70.94 | 1.12 |

Table 11: Characteristics for CLSP

Despite the small gap obtained, all the results presented in this section should be taken cautiously since optimality is not reached in most cases.

## 7. Conclusions and future research

We have used the facility location formulation of Krarup and Bilde to introduce service level constraints and costs that can distinguish between backlogs and backorders. We have considered both the ULSP and CLSP and several variants of these service level constraints. Our results show that different constraints can lead to very different solutions. We also added FIFO constraints in our problems to analyze the impact of an order management policy on the solutions. FIFO constraints emphasize the differences that exist between each service level, highlighting the importance of the definition of the service level considered. These constraints also have an impact on the average waiting time. Variable costs and fixed costs lead to changes in solutions, which do not take full advantage of the backordering and backlogging possibilities any more. Fixed backordering costs appear to have more impact on solutions than variable backordering costs. These costs are one major difference between what is done traditionally and what we have done here: instead of imposing a unit backlog cost, we imposed fixed and variable backorder costs which give us more balanced solutions in terms of the various KPIs. This contrasts with service levels which only focus on one constraint.

In future research, we want to adapt the model to take into account the presence of several customers. This will require the development of different service level constraints. We will then be able to apply priorities for some customers or add new service level considerations such as the number of customers not served on time.

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Paper : The Impact of Service Level Constraints in Deterministic Lot Sizing with Backlogging

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Highlights :

- We introduce different service levels for the deterministic lot sizing problem.
- We propose fixed and delay-dependent backorder costs.
- We introduce a first-in-first-out order management policy.
- Several KPIs are used to measure the impact of the service levels and costs.