Inventory Routing Under Stochastic Supply and Demand

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Abstract

It is well known that the integrated optimization of multiple and inter-related decisions in a supply chain can bring important benefits to companies. In this spirit, the inventory routing problem focuses on jointly optimizing inventory replenishment and vehicle routing decisions in a distribution context. In practice, the presence of uncertainty often further complicates the problem. Whereas in the literature only demand uncertainty has been studied, we address a stochastic inventory routing problem under the consideration that both the product supply and the customer demands are uncertain. We propose a two-stage stochastic programming formulation, where routing decisions are made in the first stage, while delivery quantities, inventory levels and specific recourse actions are determined in the second stage. In this context, we analyze different recourse mechanisms such as lost sales, backlogging and an additional source for the product in a capacity reservation contract setting. We provide managerial insights from the results of computational experiments using instances based on a benchmark test set. In particular, we study the response mechanisms of the optimal solutions for different levels of uncertainty and cost configurations. Furthermore, we observe that supply and demand uncertainty have different effects on the value of taking the uncertainty into account. We also study the effect of incorporating a service level. Finally, we propose a heuristic solution method which is based on the progressive hedging algorithm and provides high-quality solutions within reasonable running times for problems with a large number of scenarios.

Keywords: Inventory routing, Stochastic programming, Capacity reservation, Service level, Progressive hedging.

1. Introduction

In this paper, we address the stochastic inventory routing problem (SIRP) in the context where both the product supply and the customer demands are uncertain. We consider the basic variant of the inventory routing problem (IRP) in a one-to-many setting, in which a single central supplier has to serve the demand of multiple customers in every period of a specified time horizon. In each period, the supplier can use a fleet of vehicles to deliver the product to the customers, while minimizing the total cost of the system. Applications of the IRP and its extensions can be found in various industries such as fresh food retailing (Neves-Moreira et al., 2019; Qiu et al., 2019), the biogas industry (Fokkema et al., 2020), and the furniture industry (Miranda et al., 2018). Whereas demand uncertainty has been studied before in the context of

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the IRP, to the best of our knowledge supply uncertainty has not yet been addressed. We use a two-stage decision framework in which the routing decisions are made in the first stage while the delivery quantities, inventory levels, and specific recourse actions are determined in the second stage.

Uncertainty plays a crucial role in supply chain management given that critical input data which are required for effective planning often are not known with certainty when the plan is made, which directly impacts the quality of the decisions. Since using inaccurate information can lead to poor performance in practice, it becomes relevant to take uncertainty into account in the decision process. In the IRP context, demand (or downstream) uncertainty appears naturally because of seasonality, changes in customer preferences, and inaccurate forecasts, among others. On the other hand, supply (or upstream) uncertainty can arise from delays or shortages from the supply source or from disruptions at the supplier's production plant. In particular, supply uncertainty may play a critical role in this context given that we consider the problem variant with a single supplier. This centralization of the service implies that even relatively small supply disruptions at the supplier will affect the service to the customers. This is often disregarded in stochastic IRP studies.

Several variants of the SIRP have been studied over the past decades. Most of them (including ours) differ by the specific features that they consider, such as the random variables that are modeled, and the type of planning horizon considered, among others. We review them to put our contribution into perspective. However, given the multiple extensions of the IRP that can be explored in practice, we only review SIRPs considering demand uncertainty in a finite planning horizon for road-based transportation applications. We are not aware of any work addressing the basic variant of the IRP with supply uncertainty. For extensive reviews on the IRP, including stochastic components, we refer the reader to Andersson et al. (2010) and Coelho et al. (2014b). For studies exploring infinite horizon problems we refer the reader to the works of Jaillet et al. (2002) and Hvattum et al. (2009). Also, it is worth mentioning that supply uncertainty has been explored in other contexts such as humanitarian logistics (Moreno et al., 2018), supply chain design and planning (Zeballos et al., 2014) and source selection in manufacturing (Freeman et al., 2018).

Federgruen and Zipkin (1984) were the first to consider demand uncertainty in the IRP context. They addressed a single-period SIRP, including inventory holding, shortage and transportation costs. Federgruen et al. (1986) extended this work by considering a perishable product and incorporating spoilage costs. The authors studied two different transportation alternatives: direct deliveries performed individually to each customer, and multi-customer routes carried out by a fleet of vehicles. Huang and Lin (2010) studied a single-period multi-product SIRP with uncertain demands and stockouts. The authors assumed that the demands only become known upon the arrival of the vehicle at the customer locations and included a recourse mechanism consisting of a return trip to the depot when stockouts occur. The objective function consists of minimizing the total cost given by the sum of planned routes, the recourse costs and expected stockout costs. The authors presented a modified ant colony optimization metaheuristic. Yu et al. (2012) addressed an IRP with split deliveries and stochastic demands. The authors included service level constraints imposing (with a given probability) both demand fulfillment and maximum storage capacity usage at the customer facilities. The authors proposed a hybrid solution approach that combines the simplification of an approximate model of the problem as well as repair and local search operators.

Bertazzi et al. (2013) addressed an IRP with stochastic demands and stockouts under an order-up-to-level replenishment policy, i.e., whenever a customer is visited, the quantity delivered is such that its inventory level reaches the maximum storage capacity. The objective function minimizes the expected total cost given by the sum of the expected inventory, out-of-stock penalties and routing costs. The authors developed a dynamic programming formulation of the problem and a rollout algorithm. Bertazzi et al. (2015) addressed a similar problem but considering that the deliveries are performed using transportation procurement. Coelho et al. (2014a) addressed the IRP under the assumption of dynamic and stochastic demands. The authors proposed different heuristic policies for the problem. A single vehicle is used and transshipments between customers are allowed. The objective function consists of minimizing the sum of inventory, shortage, routing and transshipment costs. Nolz et al. (2014) addressed an IRP appearing in a medical waste collection application where demands are stochastic. In the problem, a single vehicle is used to pick up medical waste boxes from pharmacies. The authors proposed a two-stage stochastic programming formulation for the problem and a heuristic method based on an adaptive large neighborhood search algorithm. The objective function minimizes the sum of fixed and variable routing costs plus the expected second-stage cost given by the sum of excess inventory costs and penalty costs imposed for picking up less than a given threshold when visiting a pharmacy.

Gruler et al. (2018) addressed a single-period IRP with stochastic demands and stockouts. The objective function consists of minimizing the sum of expected inventory and routing costs. The authors presented a simheuristic, based on the combination of a variable neighborhood search metaheuristic with simulation. Nikzad et al. (2019) addressed a stochastic IRP appearing in medical drug distribution with uncertain demands. The objective function minimizes the sum of inventory, transportation and stockout costs. The authors presented a two-stage stochastic programming formulation and two chance-constrained stochastic formulations. A matheuristic solution algorithm is proposed. Markov et al. (2018) presented a unified framework for various classes of rich routing problems with stochastic demands, including, among others, different classes of IRPs (health care, waste collection and maritime IRP). The framework includes real-world demand forecasting techniques to provide the model with the expected demands. The authors also explicitly modeled undesirable events as well as recourse actions. Markov et al. (2020) addressed an IRP with stochastic demands appearing in a recyclable waste collection application. The authors developed an adaptive large neighborhood search algorithm and used a realistic demand forecasting model to estimate the expected demands and the uncertainty levels, as in Markov et al. (2018).

IRPs for perishable products with stochastic demands and fixed deterministic shelf-lives were studied by Soysal et al. (2015), Soysal et al. (2018) and Crama et al. (2018). In the problem addressed by Soysal et al. (2015) the objective function consists of minimizing the total cost, given by the sum of routing, inventory and waste costs. The problem allows unmet demands to be backlogged and multiple visits to the customers in each time period. The authors proposed several chance-constrained models for the problem. Soysal et al. (2018) extended this study by considering multiple perishable products and collaboration among different suppliers. They presented a chance-constrained formulation and a deterministic approximate formulation of the chance-constrained program. Crama et al. (2018) studied a problem including a maximum duration for the vehicle routes and target service levels. In their problem, the objective function consists of maximizing the expected profit given by the total sales revenue minus the acquisition, distribution, and other miscellaneous costs. The authors proposed several solution methods for the problem, namely an expected value method, a deliver-up-to-level method, a decomposition method relying on a stochastic dynamic programming model, and a decomposition-integration method.

Our study presents three main contributions. First, we introduce a two-stage stochastic programming formulation for the SIRP under uncertain supply of the product and uncertain customer demands. This formulation can be adapted to consider different recourse mechanisms, such as lost sales, backlogging and an additional supply source in a capacity reservation contract setting. We study for the first time supply uncertainty in the context of the basic variant of the IRP as well as a capacity reservation contract setting in the IRP context. As a second contribution, we present a heuristic solution method based on the progressive hedging algorithm. This method provides high-quality solutions within reasonable running times for instances with a large number of scenarios. Our final contribution consists of providing managerial insights from experiments using instances based on a benchmark test set from the literature. In particular, we study the response mechanisms of the optimal solutions for different levels of uncertainty and cost configurations. Furthermore, we observe that supply and demand uncertainty have different effects on the value of taking the uncertainty into account. We also study the effect of incorporating a service level.

The rest of the paper consists of five additional sections. Section 2, describes the problem and the mathematical notation. Section 3 introduces the formulations for the different cases of the problem, and Section 4 presents our heuristic method for the same cases. Section 5 describes the computational experiments and presents the discussion of the results. Finally, Section 6 concludes the paper.

2. Problem description

The two-stage SIRP consists of a supplier, whose depot is denoted by node 0, who has to serve the demand of N customers in each one of the T periods of a specified time horizon. In this problem, the customers are represented by the set $\mathcal{C} = \{1, \ldots, N\}$ and the time horizon by $\mathcal{T} = \{1, \ldots, T\}$. To serve the customer demands, the supplier can use up to K vehicles, each one having a capacity Q. All the vehicles are based at the depot and are represented by the set $\mathcal{K} = \{1, \ldots, K\}$. The vehicle routes take place in a distribution network represented by the set of arcs $\mathcal{A} = \{(i, j) : i, j \in \mathcal{N}, i \neq j\}$, where $\mathcal{N} = \{0\} \cup \mathcal{C}$ is the set of all the facilities of the system (supplier's depot and its customers). Every route starts from and must return to the depot. Also, a routing cost c_{ij} is incurred every time a vehicle traverses arc $(i, j) \in \mathcal{A}$. An inventory holding cost h_i^t has to be paid for every unit of product in stock at the end of each period $t \in \mathcal{T}$ in each facility $i \in \mathcal{N}$. In addition, there is an initial amount I_i^0 available in every facility $i \in \mathcal{N}$ at the beginning of the time horizon. Finally, the stock at hand at each customer $i \in \mathcal{C}$ is restricted by the maximum storage limit C_i.

The supply and demands are random variables with known discrete probability distributions (assuming independence for all facilities and periods). Let $\mathcal{S}$ denote the finite set of all the possible scenarios (supply and demand realizations), and let ρ_s be the probability of occurrence of scenario $s \in \mathcal{S}$, with $\rho_s > 0$, $\forall s \in \mathcal{S}$ and $\sum_{s \in \mathcal{S}} \rho_s = 1$. Let d_{is}^t be the demand of customer $i \in \mathcal{C}$ in time period $t \in \mathcal{T}$ under scenario $s \in \mathcal{S}$ and let r_s^t be the amount of product the supplier receives in time period $t \in \mathcal{T}$ under scenario $s \in \mathcal{S}$.

In the SIRP, any quantity can be delivered to the customers as long as the maximum holding capacity is not exceeded. In addition, we work under the following assumptions: the storage capacity of the supplier is large enough to store all the received amounts at the depot; the demand of a given time period can be satisfied with a delivery performed in the same period; and the amount the supplier receives in each period can be used to perform deliveries in that same period. We also assume that the supply and demand realizations are known before all the vehicles depart from the supplier depot. Therefore, the SIRP consists of determining, in the first stage, the vehicle routes that will be performed in each time period and, after the realization of the supply and demand scenario (second stage), the delivery quantities and the required recourse decisions (if any) such that the total cost is minimized. This total cost is given by the first-stage cost (vehicle routing cost) plus the expected cost of the second-stage decisions. Figure 1 shows the timing of the events that we assume in this paper.

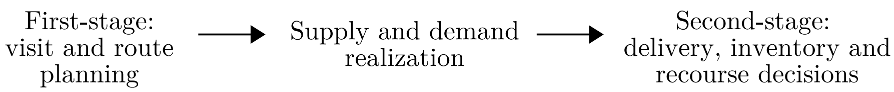


Figure 1: Timing of the events in the SIRP

Adulyasak et al. (2015) pointed out that this type of setting follows real-world practice, in which some decisions are planned beforehand using information about possible values of the input data (e.g., product availability and customer demands) and these plans remain fixed in the execution phase. This is done with the aim of designing plans that are less sensitive to data uncertainty (Vladimirou and Zenios, 1997) and also to maintain consistency in the planned activities and to avoid large disruptions in the initial plan. Particularly, consistency represents an important issue in distribution planning activities since typically the planned visits have to be informed in advance to the customers and the required resources need to be prepared (Kovacs et al., 2014; Coelho et al., 2012). It is worth mentioning that a similar timing of events has been used in other studies (Nikzad et al., 2019; Adulyasak et al., 2015; Nolz et al., 2014).

3. Two-stage stochastic programming formulations

In this section, we describe the mathematical formulations that we introduce for the SIRP. First, in Sections 3.1 and 3.2 we present formulations for the SIRP with lost sales and backlogging as recourse decisions, respectively. Then, in Section 3.3, we describe a mathematical formulation for the SIRP with a capacity reservation contract (CRC), which can be used as an additional recourse mechanism in the second stage, but also requires an additional first-stage decision.

3.1. Lost sales formulation

To model the SIRP with lost sales, we introduce the parameter a_i, which is the penalty incurred by the supplier for every unit of unmet demand at customer $i \in \mathcal{C}$ in each time period. This penalty can be interpreted as the opportunity cost for the stockouts or as the outsourcing cost paid to a third-party responsible for delivering the product to the customers. Also, we introduce the following decision variables:

$x_{ij}^{kt} \in \{0,1\}$: 1 if and only if arc (i,j) is traversed by vehicle k in period t;
$y_i^{kt} \in \{0,1\}$: 1 if and only if vehicle k visits facility i in period t;
$I_{is}^{t} \geq 0$: inventory level at facility i at the end of period t under scenario s;
$q_{is}^{kt} \geq 0$: delivery quantity for customer i by vehicle k in time period t under scenario s;
$u_{is}^{t} \geq 0$: unmet demand at customer i in period t under scenario s.

Given this notation, the formulation for the two-stage SIRP is the following:

$$\min \quad \sum_{(i,j)\in\mathcal{A}}\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} c_{ij}x_{ij}^{kt} + \sum_{s\in\mathcal{S}}\rho_s\left(\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}} h_i^t I_{is}^t + \sum_{i\in\mathcal{C}}\sum_{t\in\mathcal{T}} a_i u_{is}^t\right) \tag{1}$$

$$\text{s.t.} \quad I_{0s}^t = I_{0s}^{t-1} + r_s^t - \sum_{i\in\mathcal{C}}\sum_{k\in\mathcal{K}} q_{is}^{kt} \qquad t\in\mathcal{T}, s\in\mathcal{S}, \tag{2}$$

$$I_{is}^t = I_{is}^{t-1} + \sum_{k\in\mathcal{K}} q_{is}^{kt} + u_{is}^t - d_{is}^t \qquad i\in\mathcal{C}, t\in\mathcal{T}, s\in\mathcal{S}, \tag{3}$$

$$I_{is}^{t-1} + \sum_{k\in\mathcal{K}} q_{is}^{kt} \leq C_i \qquad i\in\mathcal{C}, t\in\mathcal{T}, s\in\mathcal{S}, \tag{4}$$

$$q_{is}^{kt} \leq \min\{Q, C_i\} y_i^{kt} \qquad i\in\mathcal{C}, k\in\mathcal{K}, t\in\mathcal{T}, s\in\mathcal{S}, \tag{5}$$

$$\sum_{i\in\mathcal{C}} q_{is}^{kt} \leq Q y_0^{kt} \qquad k\in\mathcal{K}, t\in\mathcal{T}, s\in\mathcal{S}, \tag{6}$$

$$\sum_{j\in\mathcal{N}:j\neq i} x_{ji}^{kt} = y_i^{kt} \qquad i\in\mathcal{N}, k\in\mathcal{K}, t\in\mathcal{T}, \tag{7}$$

$$\sum_{j\in\mathcal{N}:j\neq i} x_{ij}^{kt} = y_i^{kt} \qquad i\in\mathcal{N}, k\in\mathcal{K}, t\in\mathcal{T}, \tag{8}$$

$$\sum_{i\in\mathcal{B}}\sum_{j\in\mathcal{B}:j\neq i} x_{ij}^{kt} \leq \sum_{i\in\mathcal{B}} y_i^{kt} - y_\ell^{kt} \qquad \forall\mathcal{B}\subseteq\mathcal{C}, |\mathcal{B}|\geq 2, k\in\mathcal{K}, t\in\mathcal{T}, \ell\in\mathcal{B}, \tag{9}$$

$$\sum_{k\in\mathcal{K}} y_i^{kt} \leq 1 \qquad i\in\mathcal{C}, t\in\mathcal{T}, \tag{10}$$

$$I_{is}^t \geq 0 \qquad i\in\mathcal{N}, t\in\mathcal{T}, s\in\mathcal{S}, \tag{11}$$

$$q_{is}^{kt} \geq 0 \qquad i\in\mathcal{C}, k\in\mathcal{K}, t\in\mathcal{T}, s\in\mathcal{S}, \tag{12}$$

$$u_{is}^t \geq 0 \qquad i\in\mathcal{C}, t\in\mathcal{T}, s\in\mathcal{S}, \tag{13}$$

$$y_i^{kt} \in \{0,1\} \qquad i\in\mathcal{N}, k\in\mathcal{K}, t\in\mathcal{T}, \tag{14}$$

$$x_{ij}^{kt} \in \{0,1\} \qquad (i,j)\in\mathcal{A}, k\in\mathcal{K}, t\in\mathcal{T}. \tag{15}$$

The objective function (1) consists of minimizing the total cost, given by the sum of transportation costs (first-stage) and inventory holding and lost sales costs (second-stage). Constraints (2) and (3) balance the inventory at the supplier and the customers, respectively, for every scenario. Constraints (4) impose the customers' storage capacity. Constraints (5) link delivery and visit variables. Constraints (6) enforce the capacity of the vehicles in every scenario. Constraints (7) and (8) are the vehicle flow conservation while constraints (9) are the subtour elimination constraints (SECs). Constraints (10) limit the number of visits to each customer every time period.

3.2. Backlogging formulation

Instead of assuming that all the unmet demand is immediately lost, we can model a situation in which the unmet demand can be delivered in later periods with an associated penalty cost, i.e., we can use backlogging as the recourse action in the second stage. We introduce the variable B_{is}^t which is the amount backlogged at customer $i \in \mathcal{C}$ in time period $t \in \mathcal{T}$ under scenario $s \in \mathcal{S}$. Additionally, let a_i be now the unit backlogging cost at customer $i \in \mathcal{C}$ in each time period. Then, the two-stage stochastic programming formulation for the SIRP with backlogging can be stated as follows:

$$\min \quad \sum_{(i,j)\in\mathcal{A}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} c_{ij} x_{ij}^{kt} + \sum_{s\in\mathcal{S}} \rho_s \left(\sum_{i\in\mathcal{N}} \sum_{t\in\mathcal{T}} h_i^t I_{is}^t + \sum_{i\in\mathcal{C}} \sum_{t\in\mathcal{T}} a_i B_{is}^t \right) \tag{16}$$

$$\text{s.t.} \quad I_{is}^t - B_{is}^t = I_{is}^{t-1} - B_{is}^{t-1} + \sum_{k\in\mathcal{K}} q_{is}^{kt} - d_{is}^t \qquad i \in \mathcal{C}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{17}$$

$$B_{is}^t \geq 0 \qquad i \in \mathcal{C}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{18}$$

$$(2), (4)\text{-}(12) \text{ and } (14)\text{-}(15).$$

The objective function (16) consists of minimizing the total cost, given by the sum of routing costs and expected inventory holding and backlogging costs. Constraints (17) balance the inventory at the customers, now considering the backlogging variable, and constraints (18) define the domain of the new decision variable. Notice that we do not force the backlogging to be null in the last period, which can be viewed as a lost sales allowance at the end of the planning horizon, or a backlogged amount that will be carried over to the next planning horizon.

3.3. Capacity reservation contract formulation

In this section, we model the SIRP with a CRC as an additional recourse mechanism. In this setting, the supplier can make a contract with an external provider to reserve a certain amount of manufacturing capacity upfront, such that the external provider is able to produce any amount within the limits of the reserved capacity in each period (Serel et al., 2001). The external provider delivers the extra amount directly to the supplier, who subsequently transports this to the customers. When a CRC is established, the purchasing cost for the supplier is typically lower than the expected cost on the spot market (in our case, lost sales or backlogging costs). However, the supplier has to pay a certain amount upfront for this capacity reservation, irrespective of whether this capacity will be later used or not. CRCs allow the supplier to reduce purchasing costs and shortage risks, while the resource utilization of the external provider is increased (Li et al., 2020). This type of business-to-business arrangement is especially useful in uncertain environments and can be found in many industries such as commodity chemicals or semiconductor manufacturing (Kleindorfer and Wu, 2003; Serel, 2007).

In our problem, if the supplier contracts this additional capacity, it incurs a fixed reservation cost in the first stage and a variable cost for each unit of the reserved capacity that is actually used in the second stage. The contracted supplementary capacity is a multiple of a base capacity Δ offered by the external source. The extra amount that can be made available at the supplier facility at the beginning of any time period of the planning horizon is a supplementary recourse in the second stage, together with lost sales or backlogging, as in Sections 3.1 and 3.2, respectively.

To model this problem, we consider the following additional notation. Let f be a fixed reservation cost incurred for each Δ units of external capacity contracted and p be the procurement cost for each extra unit actually acquired by the supplier. We also introduce the following decision variables:

$z \in \mathbb{Z}^+$: number of times the base capacity Δ is contracted by the supplier in the first stage;

$w_s^t \geq 0$: extra amount made available at the supplier at the beginning of period t under scenario s.

Given this notation, the two-stage stochastic programming formulation for the SIRP with CRC and lost sales can be stated as follows:

$$\min \quad \sum_{(i,j)\in\mathcal{A}}\sum_{k\in\mathcal{K}}\sum_{t\in\mathcal{T}} c_{ij}x_{ij}^{kt} + fz + \sum_{s\in\mathcal{S}}\rho_s\left(\sum_{i\in\mathcal{N}}\sum_{t\in\mathcal{T}} h_i^t I_{is}^t + \sum_{i\in\mathcal{C}}\sum_{t\in\mathcal{T}} a_i u_{is}^t + \sum_{t\in\mathcal{T}} p w_s^t\right) \tag{19}$$

$$\text{s.t.} \quad I_{0s}^t = I_{0s}^{t-1} + r_s^t + w_s^t - \sum_{i\in\mathcal{C}}\sum_{k\in\mathcal{K}} q_{is}^{kt} \qquad t\in\mathcal{T}, s\in\mathcal{S}, \tag{20}$$

$$w_s^t \leq z\Delta \qquad t\in\mathcal{T}, s\in\mathcal{S}, \tag{21}$$

$$w_s^t \geq 0 \qquad t\in\mathcal{T}, s\in\mathcal{S}, \tag{22}$$

$$z\in\mathbb{Z}, \tag{23}$$

(3)-(15).

The objective function (19) consists of minimizing the total cost, given by the sum of routing and capacity reservation costs (first stage) and expected second-stage cost, given by the inventory holding and lost sales costs plus the total procurement of the extra amounts actually acquired by the supplier. Constraints (20) balance the inventory at the supplier, in which there are now two possible sources of the product: the regular source (parameter r) and the extra source (variable w). Constraints (21) allow the usage of the additional capacity in each time period up to the level actually reserved in advance (first stage) by the supplier. Constraints (22) and (23) define the domain of the usage and reservation decision variables, respectively.

Note that this model also uses the lost sales variables (u), as used in Section 3.1. The variant for this formulation considering backlogging instead of lost sales can be obtained by modifying the inventory conservation constraints of the customers to include the backlogging variable (B), as in Section 3.2. Also, it is worth mentioning that the model can be extended to a more general approach with several (different) modular capacities, e.g., L alternatives offered in base capacities $\Delta_1, \Delta_2, \ldots, \Delta_L$, of which there are $U_1, U_2, \ldots, U_L$ units available for reservation, respectively. Integer variables $z_1, z_2, \ldots, z_L$, bounded by their respective availability ($z_\ell \leq U_\ell,\ \ell = 1, \ldots, L$), can be defined to indicate the usage of the different modular alternatives. Finally, the extra amount made available in each time period (w_s^t) would be bounded by the sum of the reserved capacities, i.e., $w_s^t \leq \sum_{\ell=1}^{L} z_\ell\Delta_\ell$.

3.4. Remarks on the computational implementation

We provide a few remarks regarding the computational implementation of the proposed formulations. First of all, note that we used a complete directed graph to represent the problem, with an arc set $\mathcal{A} = \{(i,j)\colon i,j\in\mathcal{N}, i\neq j\}$. However, when the travel costs are assumed to be symmetric, i.e., $c_{ij} = c_{ji},\ \forall(i,j)\in\mathcal{A}$, the arc set can be replaced with the following edge set $\mathcal{E} = \{(i,j)\colon i,j\in\mathcal{N}, i<j\}$. This allows us to model the problem using a considerably smaller number of vehicle flow variables x, which positively impacts the performance of the solver. The constraints involving these variables must be modified accordingly.

Regarding the separation of the SECs (9), which are used in the three formulations, we use an exact procedure that relies on several minimum cut problems, which are solved using the Concorde solver (Applegate et al., 2018). For this, consider the following notation. Let $\bar{y}_i^{kt}$ and $\bar{x}_{ij}^{kt}$ represent, for a given solution (fractional or integer) found during the branch-and-cut (B&C) process, the values for the variables y_i^{kt} and x_{ij}^{kt}, respectively. The algorithm then builds a graph for every pair (k,t) with $\bar{y}_0^{kt} > 0$. The vertex set of each graph consists of all the nodes $i \in \mathcal{C}$ of the solution for which $\bar{y}_i^{kt}$ takes a positive value. The weight of the edges of each graph is set to $\bar{x}_{ij}^{kt}$, for every pair of vertices of the corresponding graph. Then, for each customer vertex of the constructed graph, the separation procedure solves a minimum $s-t$ cut problem, where the source vertex is set as the supplier node ($s = 0$) and the sink vertex is set as the customer node ($t = i$). If the minimum cut capacity is less than $2\bar{y}_i^{kt}$ then a violated SEC has been found (Adulyasak et al., 2014; Alvarez et al., 2020). The set $\mathcal{B}$ of the respective equation contains the nodes of the minimum cut and we add constraints (9) with $\ell = \arg\max_{i \in \mathcal{B}}\{\bar{y}_i^{kt}\}$, for every vehicle and time period. This separation procedure is applied only at the root node and to integer solutions, in order to work with a reduced number of cuts.

We also explored symmetry breaking constraints (SBCs) in our implementation. This is an important issue given that there can be a large number of symmetric solutions in each time period due to the homogeneous nature of the vehicle fleet, which negatively impacts the performance of the B&C algorithm. We explored the SBCs used by Adulyasak et al. (2014) and Coelho and Laporte (2013). In particular, in our implementation we used the following SBCs:

$$y_0^{kt} \leq y_0^{k-1,t} \qquad k \in \mathcal{K}\backslash\{1\}, t \in \mathcal{T}, \tag{24}$$

$$\sum_{i=1}^{j} 2^{(j-i)} y_i^{kt} \leq \sum_{i=1}^{j} 2^{(j-i)} y_i^{k-1,t} \qquad j \in \mathcal{C}, k \in \mathcal{K}\backslash\{1\}, t \in \mathcal{T}. \tag{25}$$

Constraints (24) allow the use of vehicle k only if vehicle $k-1$ is used in the same time period. Constraints (25) belong to the lexicographic ordering constraints family (Jans, 2009). These constraints assign a unique number to each possible subset of customers on a route and order the vehicles according to this number. This combination of SBCs was chosen after several preliminary experiments using a subset of the instances with two vehicles. A similar combination of SBCs was also used by Adulyasak et al. (2014).

Notice also that we can model the problem using a formulation without a vehicle index, since the fleet is considered to be homogeneous in terms of capacity and travel times and costs. However, in that case we would need either MTZ-like SECs or capacity cuts as both SECs and vehicle capacity constraints. These capacity cuts would have the form shown in (26).

$$Q \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}: j \neq i} x_{ij}^t \leq Q \sum_{i \in \mathcal{B}} y_i^t - \sum_{i \in \mathcal{B}} q_{is}^t \quad \forall \mathcal{B} \subseteq \mathcal{C}, |\mathcal{B}| \geq 2, t \in \mathcal{T}, s \in \mathcal{S}. \tag{26}$$

The former option would yield a considerably weaker formulation. On the other hand, the latter option results in capacity cuts including the delivery quantity variables, which are second-stage variables. Thus, it would be necessary to separate the capacity cuts also for each scenario, which becomes prohibitive when the number of scenarios is large.

Also notice that all the mathematical formulations presented in the previous section can be reformulated without using visit decision variables (y). Nevertheless, this could negatively affect the performance of the general-purpose optimization software used to solve the models because these variables are useful in the branching process. Branching on the visit variables y (instead of on the vehicle flow variables x) implies a focus on determining in which periods each customer is visited (instead of which arcs are used). Such a branching strategy can be very effective as this highly constrains the quantity that can be delivered in each customer visit and can significantly improve the lower bounds (LB) in many nodes of the B&C tree

(Desaulniers et al., 2016). It is worth mentioning that in our tests we used the default branching priorities on the variables of the solver.

4. A progressive hedging-based heuristic for the SIRP

This section presents the heuristic algorithm that we propose to solve the SIRP, which is based on the progressive hedging (PH) algorithm (Rockafellar and Wets, 1991). In their work, Rockafellar and Wets proposed a scenario-based decomposition method for stochastic programs based on an augmented Lagrangean strategy. The method solves a series of subproblems resulting from the scenario decomposition and guides the search to find a solution in which the aggregation of the subproblem solutions is non-anticipative (i.e., the first-stage solution is not scenario-dependent) and optimal. The authors proved that their method converges to a global optimum in the convex case and showed that if it converges in the nonconvex case when the subproblems are solved to local optimality then the resulting solution is a local optimum. Several PH-based heuristics have been proposed in the literature for stochastic problems with integer variables, e.g., Løkketangen and Woodruff (1996) for mixed integer $(0,1)$ multi-stage stochastic programs, Haugen et al. (2001) for stochastic lot-sizing problems, Crainic et al. (2011) for a stochastic network design problem and Lamghari and Dimitrakopoulos (2016) for open-pit mine production scheduling under uncertainty.

In our approach, when we apply the scenario decomposition to the original stochastic problem (Section 4.1) it results in a series of subproblems that take the form of a deterministic IRP with visiting costs for each scenario s (which we will refer to as IRP(s)). These problems are solved using an iterated local search (ILS)-based hybrid method (Section 4.4). In each outer iteration of the algorithm, the cost parameters of each scenario are adjusted to reflect the differences between the scenario solution and a reference solution (Section 4.3). These adjustments are made with the aim of reaching a consensus on the first-stage solutions over all the scenarios and thus to a feasible solution for the complete stochastic problem. The reference solution is constructed from the solution of all the scenarios in the previous iteration (Section 4.2). All the components of the overall approach are described in the upcoming sections. It is worth mentioning that in this section we describe the application of this method for the SIRP with lost sales, but the procedure can be applied analogously for both the backlogging case and the CRC case.

4.1. Scenario decomposition for the SIRP

One can see that the formulation presented in Section 3.1 has a block-angular structure (each block representing a deterministic IRP with lost sales for each scenario $s \in \mathcal{S}$) with constraints (5) and (6) linking the first- and second-stage variables. These linking constraints forbid the delivery quantities of every scenario to take positive values when the respective first-stage variables are zero. Thus, we can reformulate the problem after creating a copy of the first-stage variables for each scenario $s \in \mathcal{S}$ (x_{ijs}^{kt} and y_{is}^{kt}), as follows:

$$\min \quad \sum_{s\in\mathcal{S}} \rho_s \left(\sum_{(i,j)\in\mathcal{A}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} c_{ij} x_{ijs}^{kt} + \sum_{i\in\mathcal{N}} \sum_{t\in\mathcal{T}} h_i^t I_{is}^t + \sum_{i\in\mathcal{C}} \sum_{t\in\mathcal{T}} a_i u_{is}^t \right) \tag{27}$$

$$\text{s.t.} \quad q_{is}^{kt} \leq \min\{Q, C_i\} y_{is}^{kt} \qquad i \in \mathcal{C}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{28}$$

$$\sum_{i \in \mathcal{C}} q_{is}^{kt} \leq Q y_{0s}^{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{29}$$

$$\sum_{j \in \mathcal{N}: j \neq i} x_{jis}^{kt} = y_{is}^{kt} \qquad i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{30}$$

$$\sum_{j \in \mathcal{N}: j \neq i} x_{ijs}^{kt} = y_{is}^{kt} \qquad i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{31}$$

$$\sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{B}: j \neq i} x_{ijs}^{kt} \leq \sum_{i \in \mathcal{B}} y_{is}^{kt} - y_{\ell s}^{kt} \qquad \forall \mathcal{B} \subseteq \mathcal{C}, |\mathcal{B}| \geq 2, k \in \mathcal{K}, t \in \mathcal{T}, \ell \in \mathcal{B}, s \in \mathcal{S}, \tag{32}$$

$$\sum_{k \in \mathcal{K}} y_{is}^{kt} \leq 1 \qquad i \in \mathcal{C}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{33}$$

$$y_{is}^{kt} = \hat{y}_i^{kt} \qquad i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{34}$$

$$x_{ijs}^{kt} = \hat{x}_{ij}^{kt} \qquad (i,j) \in \mathcal{A}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{35}$$

$$y_{is}^{kt} \in \{0,1\} \qquad i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{36}$$

$$x_{ijs}^{kt} \in \{0,1\} \qquad (i,j) \in \mathcal{A}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{37}$$

$$\hat{y}_i^{kt} \in \{0,1\} \qquad i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}, \tag{38}$$

$$\hat{x}_{ij}^{kt} \in \{0,1\} \qquad (i,j) \in \mathcal{A}, k \in \mathcal{K}, t \in \mathcal{T}, \tag{39}$$

(2)-(4) and (11)-(13).

Constraints (34) and (35) ensure that the first-stage solutions will be the same for all the scenarios. These constraints are imposed to guarantee that a single "implementable" solution will be obtained (Rockafellar and Wets, 1991), i.e., a single set of vehicle routes (and their respective visit decisions) for each time period over all the scenarios, instead of scenario-tailored first-stage solutions. The variables $\hat{y}_i^{kt}$ and $\hat{x}_{ij}^{kt}$ are referred to as the "overall" first-stage variables (Crainic et al., 2011).

Following the separation procedure of the PH algorithm, constraints (34) and (35) are relaxed using an augmented Lagrangean method, alternatively referred to as multiplier method (Luenberger and Ye, 2016), which results in the following objective function for the formulation:

$$\begin{aligned}\min \quad & \sum_{s \in \mathcal{S}} \rho_s \Bigg(\sum_{(i,j) \in \mathcal{A}} \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} c_{ij} x_{ijs}^{kt} + \sum_{i \in \mathcal{N}} \sum_{t \in \mathcal{T}} h_i^t I_{is}^t + \sum_{i \in \mathcal{C}} \sum_{t \in \mathcal{T}} a_i u_{is}^t + \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \lambda_{is}^{kt} (y_{is}^{kt} - \hat{y}_i^{kt}) \\ & + \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \delta (y_{is}^{kt} - \hat{y}_i^{kt})^2 + \sum_{(i,j) \in \mathcal{A}} \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \mu_{ijs}^{kt} (x_{ijs}^{kt} - \hat{x}_{ij}^{kt}) + \frac{1}{2} \sum_{(i,j) \in \mathcal{A}} \sum_{k \in \mathcal{K}} \sum_{t \in \mathcal{T}} \delta (x_{ijs}^{kt} - \hat{x}_{ij}^{kt})^2 \Bigg), \end{aligned} \tag{40}$$

with unrestricted Lagrangean multipliers λ_{is}^{kt} and μ_{ijs}^{kt} for the relaxed constraints (34) and (35), respectively, and a penalty term δ. They penalize the difference of the values of the visit and routing decisions between the scenario solution (y_{is}^{kt} and x_{ijs}^{kt}) and the "overall" first-stage variables ($\hat{y}_i^{kt}$ and $\hat{x}_{ij}^{kt}$). Then, given that the variables x_{ijs}^{kt} and y_{is}^{kt} are binary, the function can be reduced as follows:

$$\begin{aligned} \min \quad & \sum_{s \in \mathcal{S}} \rho_s \Bigg(\sum_{(i,j)\in\mathcal{A}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} (c_{ij} + \mu_{ijs}^{kt} + \frac{1}{2}\delta - \delta\hat{x}_{ij}^{kt}) x_{ijs}^{kt} + \sum_{i\in\mathcal{N}} \sum_{t\in\mathcal{T}} h_i^t I_{is}^t + \sum_{i\in\mathcal{C}} \sum_{t\in\mathcal{T}} a_i u_{is}^t \\ & + \sum_{i\in\mathcal{N}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} (\lambda_{is}^{kt} + \frac{1}{2}\delta - \delta\hat{y}_i^{kt}) y_{is}^{kt} \Bigg) + \theta, \end{aligned} \tag{41}$$

where

$$\theta = \sum_{s\in\mathcal{S}} \rho_s \left(\frac{1}{2} \sum_{i\in\mathcal{N}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \delta(\hat{y}_i^{kt})^2 - \sum_{i\in\mathcal{N}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \lambda_{is}^{kt} \hat{y}_i^{kt} + \frac{1}{2} \sum_{(i,j)\in\mathcal{A}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \delta(\hat{x}_{ij}^{kt})^2 - \sum_{(i,j)\in\mathcal{A}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \mu_{ijs}^{kt} \hat{x}_{ij}^{kt} \right). \tag{42}$$

Notice that for a given solution for the variables $\hat{y}_i^{kt}$ and $\hat{x}_{ij}^{kt}$, the relaxed formulation decomposes by scenario. Thus, for each scenario $s \in \mathcal{S}$, the subproblem takes the form of a deterministic IRP with lost sales and visiting costs, as follows:

$$\begin{aligned} \min \quad & \sum_{(i,j)\in\mathcal{A}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \bar{c}_{ijs}^{kt} x_{ijs}^{kt} + \sum_{i\in\mathcal{N}} \sum_{t\in\mathcal{T}} h_i^t I_{is}^t + \sum_{i\in\mathcal{C}} \sum_{t\in\mathcal{T}} a_i u_{is}^t + \sum_{i\in\mathcal{N}} \sum_{k\in\mathcal{K}} \sum_{t\in\mathcal{T}} \bar{b}_{is}^{kt} y_{is}^{kt} \\ \text{s.t.} \quad & (2)\text{-}(4), (11)\text{-}(13), (28)\text{-}(33) \text{ and } (36)\text{-}(37), \end{aligned} \tag{43}$$

where $\bar{c}_{ijs}^{kt} = c_{ij} + \mu_{ijs}^{kt} + \frac{1}{2}\delta - \delta\hat{x}_{ij}^{kt}$ and $\bar{b}_{is}^{kt} = \lambda_{is}^{kt} + \frac{1}{2}\delta - \delta\hat{y}_i^{kt}$ are the routing and visiting costs of the scenario subproblem, respectively.

To devise a PH-based solution method from the previously applied decomposition, we must define a procedure to set the reference solution ($\hat{y}_i^{kt}$ and $\hat{x}_{ij}^{kt}$) as well as a procedure to guide the scenario solutions to a consensus among the first-stage solutions. These procedures are described in the upcoming sections.

4.2. Setting the reference solution

In this phase of the method, we use the first-stage solution of each scenario to identify a global trend among them. In our heuristic, we use an aggregation operator that combines the first-stage solutions over all the scenarios into a single solution by taking the weighted sum of each first-stage variable, where the weights are defined by the probability of occurrence of each scenario. This type of aggregation operator was originally proposed by Rockafellar and Wets (1991) and later used by Crainic et al. (2011) and Lamghari and Dimitrakopoulos (2016). Let v define the index of the outer iterations of the PH-based heuristic method. Then, the value of the reference solution variables $\hat{y}_i^{ktv}$ and $\hat{x}_{ij}^{ktv}$ in the iteration v of the algorithm, are obtained using equations (44) and (45), respectively, as follows:

$$\hat{y}_i^{ktv} = \sum_{s\in\mathcal{S}} \rho_s \bar{y}_{is}^{ktv} \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}, \tag{44}$$

$$\hat{x}_{ij}^{ktv} = \sum_{s\in\mathcal{S}} \rho_s \bar{x}_{ijs}^{ktv} \qquad \forall (i,j) \in \mathcal{A}, k \in \mathcal{K}, t \in \mathcal{T}, \tag{45}$$

where $\bar{y}_{is}^{ktv}$ and $\bar{x}_{ijs}^{ktv}$ are the values of the first-stage variables of the solution of scenario $s \in \mathcal{S}$ in the v-th iteration of the algorithm.

Notice that when we obtain $\hat{y}_i^{ktv}(\hat{x}_{ij}^{ktv}) \in \{0,1\}$ for a given node $i \in \mathcal{N}$ (arc $(i,j) \in \mathcal{A}$), vehicle $k \in \mathcal{K}$ and time period $t \in \mathcal{T}$, it means that we have reached a consensus on the values of the variables y_{is}^{ktv} (x_{ijs}^{ktv}) over all the scenarios in iteration v. Then, if a consensus is obtained for all the first-stage variables, the current set of solutions (one for each scenario) composes a feasible solution for the complete stochastic program. However, most of the time this is not the case and we have $0 < \hat{y}_i^{ktv} < 1$ or $0 < \hat{x}_{ij}^{ktv} < 1$, implying that the current reference solution is infeasible given the integrality requirements of the first-stage variables. Still, the values of the reference solution can be used to indicate the tendency to visit a customer or the usage of an arc by a given vehicle in a defined time period. For a given node (arc), vehicle and period, values of $\hat{y}_i^{ktv}$ $(\hat{x}_{ij}^{ktv})$ close to one indicate that the node (arc) is being visited (traversed) in the period by that vehicle in most of the scenario solutions. Analogously, a value close to zero indicates a tendency toward not visiting the node (traversing the arc).

4.3. Adjustment strategy

In each iteration of the PH-based heuristic, it is necessary to adjust the scenario subproblem costs with the aim of leading to a gradual consensus of the first-stage solutions over all the scenario subproblems and, as a consequence, of the reference solution variables. For this, different strategies can be used such as updating the multipliers λ_{is}^{kt} and μ_{ijs}^{kt} that appear in the routing and visiting costs of the objective function (43) of the scenario subproblems, using the augmented Lagrangean method (Bertsekas, 1982; Luenberger and Ye, 2016). Another strategy that can be applied is to use a heuristic rule in each iteration to directly modify the routing and visiting costs ($\bar{c}_{ijs}^{kt}$ and $\bar{b}_{is}^{kt}$, respectively) of the scenarios, instead of the multipliers (Crainic et al., 2011). In our implementation we used a heuristic strategy since it resulted in slightly better results in most of the cases we tested. Two types of heuristic adjustments are applied, namely, a global adjustment to guide the overall search and a local adjustment to influence the search at each scenario subproblem.

Given a reference solution $\hat{y}_i^{ktv}$ and $\hat{x}_{ij}^{ktv}$ in iteration v, the global adjustment tries to identify trends among the scenario solutions and then sets the costs accordingly. When a low value of $\hat{y}_i^{ktv}$ $(\hat{x}_{ij}^{ktv})$ is reached, it means that in most of the scenario solutions in iteration v, the node i is not visited (the arc (i,j) is not traversed) by vehicle k in time period t, while a large value of $\hat{y}_i^{ktv}$ $(\hat{x}_{ij}^{ktv})$ indicates the opposite. Thus, when the value of $\hat{y}_i^{ktv}$ $(\hat{x}_{ij}^{ktv})$ is less than a given parameter ε^y $(\varepsilon^x) \in (0, 0.5)$ we increase the value of the visit (travel) cost with the aim of discouraging the visit to the customer (the use of the arc) in the next iteration by all the scenarios. Similarly, when the value of $\hat{y}_i^{ktv}$ $(\hat{x}_{ij}^{ktv})$ is greater than $1 - \varepsilon^y$ $(1 - \varepsilon^x)$ we reduce the corresponding parameter so that the visit (usage of the arc) is encouraged in the scenario solutions. This strategy, for the visit and travel costs, is defined in equations (46) and (47), respectively:

$$\bar{b}_i^{ktv} = \begin{cases} \beta \bar{b}_i^{k,t,v-1} & \text{if } \hat{y}_i^{k,t,v-1} < \varepsilon^y, \\ \frac{1}{\beta} \bar{b}_i^{k,t,v-1} & \text{if } \hat{y}_i^{k,t,v-1} > 1 - \varepsilon^y, \\ \bar{b}_i^{k,t,v-1} & \text{otherwise;} \end{cases} \tag{46}$$

$$\bar{c}_{ij}^{ktv} = \begin{cases} \beta \bar{c}_{ij}^{k,t,v-1} & \text{if } \hat{x}_{ij}^{k,t,v-1} < \varepsilon^x, \\ \frac{1}{\beta} \bar{c}_{ij}^{k,t,v-1} & \text{if } \hat{x}_{ij}^{k,t,v-1} > 1 - \varepsilon^x, \\ \bar{c}_{ij}^{k,t,v-1} & \text{otherwise,} \end{cases} \tag{47}$$

with $\beta > 1$, where β is the adjustment rate of the costs.

The local adjustment strategy is applied at the level of each scenario $s \in \mathcal{S}$. We try to identify variables of the scenario solution for which there are large differences w.r.t. the current reference solution and adjust their costs, using equations (48) and (49) for the visit and routing costs, respectively:

$$\bar{b}_{is}^{ktv} = \begin{cases} \beta \bar{b}_i^{ktv} & \text{if } |\bar{y}_{is}^{k,t,v-1} - \hat{y}_i^{k,t,v-1}| > M^y \text{ and } \bar{y}_{is}^{k,t,v-1} = 1, \\ \frac{1}{\beta} \bar{b}_i^{ktv} & \text{if } |\bar{y}_{is}^{k,t,v-1} - \hat{y}_i^{k,t,v-1}| > M^y \text{ and } \bar{y}_{is}^{k,t,v-1} = 0, \\ \bar{b}_i^{ktv} & \text{otherwise;} \end{cases} \tag{48}$$

$$\bar{c}_{ijs}^{ktv} = \begin{cases} \beta \bar{c}_{ij}^{ktv} & \text{if } |\bar{x}_{ijs}^{k,t,v-1} - \hat{x}_{ij}^{k,t,v-1}| > M^x \text{ and } \bar{x}_{ijs}^{k,t,v-1} = 1, \\ \frac{1}{\beta} \bar{c}_{ij}^{ktv} & \text{if } |\bar{x}_{ijs}^{k,t,v-1} - \hat{x}_{ij}^{k,t,v-1}| > M^x \text{ and } \bar{x}_{ijs}^{k,t,v-1} = 0, \\ \bar{c}_{ij}^{ktv} & \text{otherwise,} \end{cases} \tag{49}$$

where the parameters $M^y \in (0,1)$ and $M^x \in (0,1)$ represent the threshold defining when a local adjustment has to be applied for the visit and routing variables, respectively.

4.4. Solving the IRP(s)

In our solution approach, for each scenario and iteration we have to solve a subproblem corresponding to a deterministic IRP with visiting costs. As pointed out by Crainic et al. (2011), it is not necessary to solve the scenario subproblems to optimality since we are using the PH algorithm as a heuristic procedure. Thus, to solve these problems we use an ILS-based hybrid heuristic, which has been successfully applied to solve other variants of the IRP (Alvarez et al., 2018, 2020). In this method, several components manage the different decisions of the problem. First, the local search heuristic of the method is responsible for the improvement of the routing decisions. This is done using a randomized variable neighborhood descent heuristic. Secondly, a multi-operator procedure, which is used in the perturbation mechanism, handles the visit decisions. This procedure modifies several parts of the input solution every time it is applied. In addition, the method uses a linear programming (LP) model to compute the optimal values of the delivery, inventory and recourse decisions. The input required by this model is a solution given by a set of visit decisions.

4.5. Obtaining a feasible solution for the SIRP

Once every scenario subproblem has been solved, we can use their solutions to obtain a feasible solution for the complete problem. For this, notice that we can have three cases for the solutions of the scenario subproblems in iteration v, as follows:

(i) We have a consensus on all the first-stage solutions over all the scenarios, i.e., $\hat{y}_i^{ktv}(\hat{x}_{ij}^{ktv}) \in \{0,1\}$, $\forall i \in \mathcal{N}((i,j) \in \mathcal{A}), k \in \mathcal{K}, t \in \mathcal{T}$;

(ii) We have a consensus on the visit variables over all the scenarios but we have not reached a consensus on the vehicle flow variables, i.e., $\hat{y}_i^{ktv} \in \{0,1\}$, $\forall i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}$ and $0 < \hat{x}_{ij}^{ktv} < 1$, for some $(i,j) \in \mathcal{A}, k \in \mathcal{K}, t \in \mathcal{T}$;

(iii) We have not reached a consensus neither on the visit variables nor on the vehicle flow variables, i.e., $0 < \hat{y}_i^{ktv} < 1$, for some $i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}$.

In the first case, we have obtained a feasible solution for the complete multi-scenario problem where the first-stage solution corresponds to the reference solution, i.e., $y_i^{kt} = \hat{y}_i^{ktv}$ and $x_{ij}^{kt} = \hat{x}_{ij}^{ktv}$, $\forall k \in \mathcal{K}, t \in \mathcal{T}, i \in \mathcal{N}$ and $(i,j) \in \mathcal{A}$, respectively. In the second case, the scenario solutions visit the same customers using the same vehicles in the same time periods but in different orders (using different vehicle routes). In this case, we can take the scenario solution with the lowest first-stage cost and use the values of its first-stage solution ($\bar{y}$ and $\bar{x}$) to obtain a feasible solution for the complete problem by solving the following LP model:

$$\min \quad \sum_{s\in\mathcal{S}} \rho_s \left(\sum_{i\in\mathcal{N}} \sum_{t\in\mathcal{T}} h_i^t I_{is}^t + \sum_{i\in\mathcal{C}} \sum_{t\in\mathcal{T}} a_i u_{is}^t \right) \tag{50}$$

$$\text{s.t.} \quad q_{is}^{kt} \leq \min\{Q, C_i\} \bar{y}_i^{kt} \qquad i \in \mathcal{C}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{51}$$

$$\sum_{i\in\mathcal{C}} q_{is}^{kt} \leq Q \bar{y}_0^{kt} \qquad k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}, \tag{52}$$

$$(2)\text{-}(4) \text{ and } (11)\text{-}(13),$$

where $\bar{y}_0^{kt} = 1$ indicates that vehicle k is used in time period t and $\bar{y}_i^{kt} = 1$ indicates that vehicle k visits customer i in time period t. The objective function (50) consists of minimizing the total cost, given by the sum of inventory holding and penalty costs. Constraints (51) link delivery and visit variables. Finally, constraints (52) impose the capacity of each vehicle.

In the last case, when there is no consensus on the visit variables, and consequently on the vehicle flow variables, we can still use a scenario first-stage solution to obtain a feasible solution for the whole problem. In our implementation we use the solution of scenario $\bar{s}$, such that, $\bar{s} = \arg\max_{s\in\mathcal{S}}\{\sum_{i\in\mathcal{C}} \bar{a}_i \sum_{t\in\mathcal{T}} \sum_{k\in\mathcal{K}} \bar{y}_{is}^{kt}\}$, where $\bar{a}_i = a_i / \max_{j\in\mathcal{C}}\{a_j\}$, i.e., we take the solution with the largest value of the weighted number of visits. Using the first-stage solution of $\bar{s}$ ($\bar{y}$ and $\bar{x}$) we can obtain a feasible solution for the complete problem by solving the same LP as described for the second case. It is worth mentioning that for this case we tried different strategies for the selection of the first-stage scenario solution used as basis to generate a solution for the multi-scenario problem and the above mentioned criterion led to slightly better results.

In addition, we try to improve the solution found by applying an ILS-based hybrid method similar to the one described in Section 4.4 but in this case for the multi-scenario problem. Thus, the local search and perturbation components are the same as before, and the LP model is replaced by a multi-scenario LP model.

4.6. Description of the complete heuristic

Given the components described in the previous sections, we can now describe the general structure of the solution method, whose pseudo-code is shown in Algorithm 1. Each outer iteration corresponds to an iteration of the main loop (lines 3 to 16). In the first outer iteration of the method (line 4), we set the initial values of the scenario subproblem costs (lines 5 to 6). For subsequent iterations, the required global and local adjustment strategies are applied (lines 8 and 9, respectively) at the beginning of the outer iterations. After that, we solve every scenario subproblem (line 11) and compute the updated reference solution of the iteration (line 12 and 13). At the end of every outer iteration, we construct a feasible solution for the problem (line 14) and update the best feasible solution if a new one was found. In the second outer iteration of the heuristic ($v = 1$), the visiting costs are set to an initial value b_0 in the local adjustment strategy (line 9), i.e., $b_{is}^{ktv} = b_0, \ \forall i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}$ when $v = 1$. This is done because in the original version of the problem (before the scenario decomposition) there are no visiting costs. The algorithm stops when either a consensus is reached over all the first-stage variables, when it reaches the maximum number of outer iterations, or when the running time limit is exceeded (line 3).

Algorithm 1: PH-based heuristic

1 **begin**
2 $v = 0$;
3 **while** *stopping criterion is not met* **do**
4 **if** $v = 0$ **then**
5 $\bar{b}_{is}^{kt0} = 0 \quad \forall i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}$;
6 $\bar{c}_{ijs}^{kt0} = c_{ij} \quad \forall (i,j) \in \mathcal{A}, k \in \mathcal{K}, t \in \mathcal{T}, s \in \mathcal{S}$;
7 **else**
8 Apply global adjustment strategy;
9 Apply local adjustment strategy $\forall s \in \mathcal{S}$;
10 **end**
11 **for** $s \in \mathcal{S}$ **do** Solve IRP(s) ;
12 $\hat{y}_i^{ktv} = \sum_{s \in \mathcal{S}} \rho_s \bar{y}_{is}^{ktv} \qquad \forall i \in \mathcal{N}, k \in \mathcal{K}, t \in \mathcal{T}$;
13 $\hat{x}_{ij}^{ktv} = \sum_{s \in \mathcal{S}} \rho_s \bar{x}_{ijs}^{ktv} \qquad \forall (i,j) \in \mathcal{A}, k \in \mathcal{K}, t \in \mathcal{T}$;
14 Apply feasible solution generation procedure;
15 $v = v + 1$;
16 **end**
17 **end**

5. Computational experiments

In this section, we report the results obtained with the formulations and the heuristic algorithm previously presented. The algorithms were coded in C++ and run on a 2.1 GHz AMD Opteron 6172 processor with one thread and a limit of 18 GB of RAM. We used CPLEX v12.8 as solver. CPLEX is used in its default configuration only turning off its parallel processing and setting the tolerance to optimality to a value of 10^{-6}.

5.1. Test instances

To test our algorithms, we generated problem instances based on the benchmark set proposed by Archetti et al. (2007) for the deterministic basic variant of the IRP with a single vehicle. The original set is divided into four sets: H3, L3, H6, and L6, where L (H) stands for low (high) inventory holding costs while the digit (3 or 6) indicates the number of time periods for the instances of the set. For our experiments, we used sets H3 and H6 only as these instances provide a more significant trade-off between routing and inventory holding costs. Also, we created a new set (H9) from the instances of set H6 by extending the planning horizon of these instances to nine time periods (instead of six). It is worth mentioning that in the instances of Archetti et al. (2007), the value of the customer demands ($\bar{d}_i^t$) and the amounts made available at the supplier ($\bar{r}^t$) are constant over the planning horizon, i.e., $\bar{d}_i^t = \bar{d}_i$, $\forall i \in \mathcal{C}$, $t \in \mathcal{T}$ and $\bar{r}^t = \bar{r}$, $\forall t \in \mathcal{T}$, respectively.

We changed some parameter values of the instances with the aim of increasing the sensitivity to the random variables. Specifically, we reduced the amounts received by the supplier in each period by 25%, set the initial inventory at each customer to $\bar{I}_i^0 = 0.1T\bar{d}_i,\ \forall i \in \mathcal{C}$, and set the initial inventory at the supplier to $\bar{I}_0^0 = 0.1T\sum_{i\in\mathcal{C}} \bar{d}_i$. These changes were performed given that, for some customers, the original value of the initial inventory was large enough to serve the demand for up to two time periods, whereas the supplier initial inventory was large enough to cover all the customer demands, concealing the impact of the uncertain parameters. In addition, we multiplied the inventory holding costs by 5 to increase the relative importance of the inventory management on the total cost.

The supply and demand were assumed to be independent random variables and the scenarios were generated using a discrete uniform distribution applying a Monte Carlo simulation. For the supply, we used the range $[\bar{r}^t(1-\epsilon^r), \bar{r}^t(1+\epsilon^r)],\ \forall t \in \mathcal{T}$, where $\epsilon^r \in [0,1]$ is the supply uncertainty level. For the demand, we used the range $[\bar{d}_i^t(1-\epsilon^d), \bar{d}_i^t(1+\epsilon^d)],\ \forall i \in \mathcal{C}, t \in \mathcal{T}$, where $\epsilon^d \in [0,1]$ is the demand uncertainty level. We set the probability of occurrence of each scenario as $\rho_s = 1/S,\ \forall s \in \mathcal{S}$, i.e., all the scenarios have the same probability of occurrence.

It is worth highlighting that, with the changes applied to the instance parameters and the assumed distributions for the uncertain parameters, the expected demand coverage is 95%, i.e., the expected value of the sum of the amounts made available at the supplier plus the initial inventories ($\mathbb{E}[\sum_{t\in\mathcal{T}} r^t] + I_0^0 + \sum_{i\in\mathcal{C}} I_i^0$) equals $0.95D$, where D is the expected value of the sum of the demands of all the customers over the planning horizon ($D = \mathbb{E}[\sum_{i\in\mathcal{C}}\sum_{t\in\mathcal{T}} d_i^t]$). This applies for all the instances, independent of the length of the planning horizon.

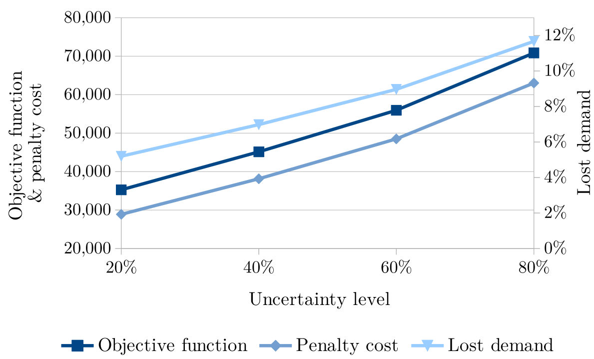
The values of the penalty terms were set as $a_i = \lceil \hat{a}(F + 2c_{0i}/m^c) \rceil$, with $m^c = \max_{i\in\mathcal{C}}\{c_{0i}\}$ and where $\hat{a}$ is a predefined penalty level and F is a fixed penalty value, as in Adulyasak et al. (2015). Unless stated otherwise, we use $\hat{a} = 4$ and $F = 50$ for the lost sales and backlogging penalties and $\Delta = 0.1\bar{r}$, $f = 5{,}000$ and $p = \hat{a}F/2$ for the CRC parameters.

In our experiments, we considered the instances with up to 30 customers. Travel costs are computed as Euclidean distances and then rounded to the nearest integer. We use one and two vehicles, dividing the vehicle capacity of the original instance by the number of vehicles and then rounding to the nearest integer.

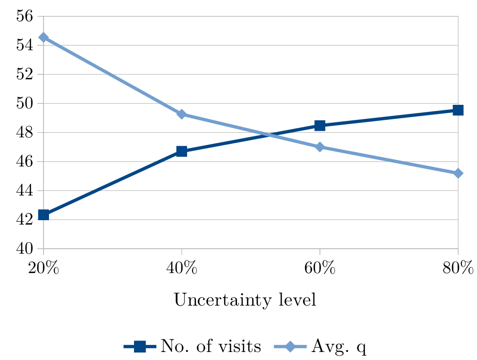
5.2. Results with the lost sales formulation

This section shows the results of the formulation with lost sales as recourse, presented in Section 3.1. We imposed a maximum running time of two hours to solve each instance with the formulation. We solved all the instances with 100, 200 and 500 scenarios and for increasing values of the uncertainty levels ϵ^r and ϵ^d, from 0.2 to 0.8 (increasing both supply and demand uncertainty by 0.2), but only show the results for those instances solved to optimality within the time limit. Figure 2a shows the average objective function value, penalty cost and percentage of lost demand of all the optimal solutions, and Figure 2b indicates the average total number of visits (No. of visits) and delivery size (Avg. q). In addition, Table 1 shows the average time (Time) in seconds required to solve all the instances with the given uncertainty levels ($\epsilon^r = \epsilon^d$), the value of the stochastic solution (VSS/OF) and the expected value of perfect information (EVPI/OF) (Birge and Louveaux, 2011) w.r.t. the objective function value. In the table, columns 1 to 3 display the number of time periods (T), vehicles (K) and scenarios (S) of the instances, respectively. Column 4 (#) shows the number of instances solved to optimality (out of 30) for the instances with the sizes given in columns 1 to 3. Each cell shows the average value of the respective column header over all the instances solved to optimality (#). The results include only the instances solved to optimality in all the four cases of the uncertainty levels (i.e., $\epsilon^r = \epsilon^d = 0.2, 0.4, 0.6, 0.8$), in order to be able to compare the results.

VSS is used to compare the stochastic solution with an expected value approach. It measures the potential gains when solving the stochastic problem instead of a simple expected value problem (EV), in which the random variables are replaced by their expected values in a single scenario approach. To compute the VSS, one has to solve a problem calculating the expected result of using the EV solution (EEV), which is the same stochastic programming model in which its first-stage variables are fixed to the values of the solution of the EV problem. VSS is computed as the difference between the optimal values of the EEV problem and the stochastic programming model (RP), i.e., VSS = EEV − RP. EVPI is used to compare the stochastic solution with the wait-and-see approach, in order to provide a measure of how good a solution would be under perfect information. EVPI is computed as the difference between the optimal values of the stochastic programming model and the expected wait-and-see solution (WS), i.e., EVPI = RP − WS, where WS $= \sum_{s\in\mathcal{S}} \rho_s W^*(s)$ and $W^*(s)$ is the optimal value of the single-scenario deterministic problem associated with scenario $s \in \mathcal{S}$. We set a time limit of two hours to compute the VSS and EVPI values.



(a) Objective function, penalty cost and lost demand



(b) Number of visits and average delivery size

Figure 2: Behavior of the solutions for the lost sales case for increasing uncertainty levels

Instance set				Time (secs)				VSS/OF (%)				EVPI/OF (%)			
T	K	S	#	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8	0.2	0.4	0.6	0.8
3	1	100	30	79.46	14.28	7.77	5.94	21.22	30.28	29.00	29.19	1.89	2.02	2.11	1.80
6	1	100	20	1,388.16	100.66	37.12	16.65	16.31	27.92	36.88	36.99	2.39	2.87	2.81	2.56
9	1	100	7	919.72	115.09	48.39	17.43	13.03	28.93	33.89	41.76	2.70	4.63	4.05	3.96
3	2	100	19	1,145.68	210.43	183.52	160.94	24.10	31.35	30.39	29.08	2.78	3.70	3.74	3.64
6	2	100	5	1,950.46	550.97	465.71	321.28	14.89	30.00	33.08	34.35	2.90	4.71	4.95	5.13
3	1	200	30	258.48	31.20	19.21	12.51	21.52	29.90	29.09	29.19	1.92	2.16	2.12	1.86
6	1	200	16	2,093.92	222.19	81.76	39.53	16.47	27.41	35.30	37.54	2.49	2.95	3.20	3.09
9	1	200	6	3,160.64	468.82	132.81	52.77	10.65	28.23	32.50	38.69	2.61	4.12	4.21	4.10
3	2	200	15	1,368.06	594.36	322.03	185.53	21.77	26.21	30.20	29.06	3.18	4.09	4.18	3.90
6	2	200	2	5,566.76	1,397.09	593.60	205.71	17.44	26.53	35.95	38.46	3.71	4.99	5.53	5.67
3	1	500	30	1,383.54	196.65	87.28	51.42	21.41	29.78	30.38	29.06	1.70	1.91	1.85	1.64
6	1	500	6	1,531.68	362.76	127.61	68.47	12.02	22.10	27.54	28.13	2.78	3.89	3.61	3.62
3	2	500	8	1,753.16	1,273.75	1,026.23	756.35	25.44	27.03	27.03	28.83	2.96	4.50	4.35	4.22
Avg.			194	1,158.26	243.71	141.35	91.86	19.79	28.91	31.12	31.57	2.29	2.87	2.88	2.68

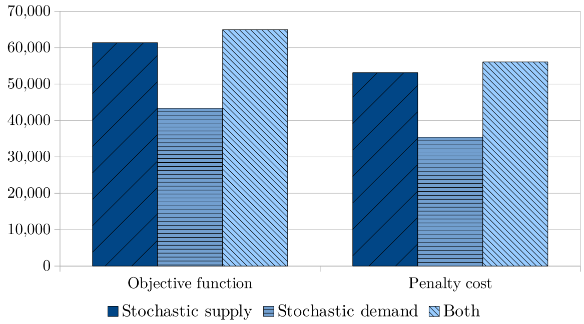
Table 1: Results for the lost sales formulation for increasing values for ϵ^r and ϵ^d

As expected, these results show that increasing uncertainty levels results in larger values of the objective function, as a consequence of the larger values of the lost demand. Also, when we increase the uncertainty levels, the solutions tend to perform more visits and the delivery sizes tend to decrease. This can be viewed as a protection mechanism against the uncertainties of the problem, which also results in larger values of the total transportation cost. Increasing uncertainty levels also resulted in larger relative VSS. For uncertainty levels of 20% ($\epsilon^r = \epsilon^d = 0.2$), the average VSS represents approximately 20% of the objective function value while for 80% of uncertainty levels ($\epsilon^r = \epsilon^d = 0.8$) it represents more than 30% of this value. These results justify the use of the stochastic programming model. Regarding the EVPI, notice that they are relatively small when compared to the objective function value. This implies that, even if the supplier had a perfect forecast for the supply and demand, this would not result in significantly better solutions, which shows the robustness of the solutions obtained with the stochastic model.

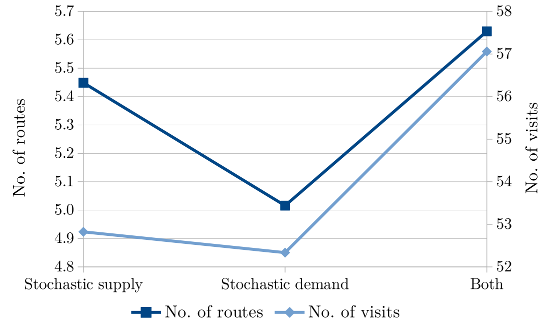
From these results, it is also possible to observe the difficulty of solving the formulation using a general-purpose optimization software, given that only 194 of the considered instances (540 in total) were solved to optimality for the four cases of the uncertainty levels. This difficulty increases especially with growing values of the number of time periods (T) and vehicles (K). Notice also that the time the solver takes to solve the instances decreases with increasing values of the uncertainty levels. This can be partially explained by the fact that dominant scenarios (i.e., scenarios with low supply and large customer demands) are more likely to appear with larger uncertainty levels. These scenarios condition the first-stage decisions since the lost sales costs are generally high. Thus, the optimal solutions will tend to avoid out-of-stock situations by planning more visits even though the product might not be available at the supplier. This is related to the protection mechanism previously mentioned. It is worth remembering that the number displayed in column '#' corresponds to the number of instances solved to optimality in all the four cases of the uncertainty levels. Notice also that the table displays fewer cases (13 rows) than the total explored ($18 = 3 \times 2 \times 3$) for the combinations of the values for T, K and S, since for some combinations the solver could not solve to optimality any of the instances for the four explored uncertainty level values.

In a different analysis, for the solutions of the instances that were solved optimally within the time limit when considering uncertainty in both supply and demand ($\epsilon^r = \epsilon^d = 0.6$), only in supply ($\epsilon^r = 0.6, \epsilon^d = 0.0$), and only in demand ($\epsilon^r = 0.0, \epsilon^d = 0.6$), Figure 3a shows the average objective function value and total penalty cost, Figure 3b displays the average total number of routes (No. of routes) and visits (No. of visits), and Figure 3c indicates the average VSS and EVPI w.r.t. the objective function value. It can be observed that smaller values of the objective function and penalty cost are obtained when we consider either only uncertain supply or only uncertain demand, when compared to the case when both are uncertain (Figure 3a). This is a result of the reduced number of uncertain parameters in the former cases. The results also show that supply uncertainty has a larger impact on the lost sales values, compared to the case with only demand uncertainty. This may be due to the pooling effect of the demand uncertainty for each customer within each time period, i.e., the variability of the demand in each time period is partially canceled among the demands themselves. This results in a larger number of routes and visits for the case when only the product supply is uncertain compared to the case when only the demands are uncertain (Figure 3b). Again, large relative VSS justify the use of the stochastic programming model even for the cases in which only one of the parameters is uncertain. Notice that, analogously to the previous experiment (with increasing uncertainty levels), the relative EVPI is relatively small, showing the robustness of the solutions of the stochastic model (Figure 3c).

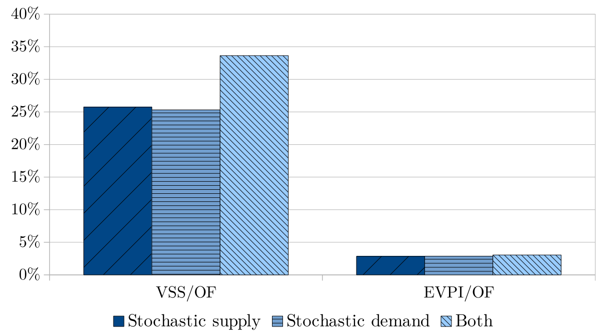
Figure 4 displays the average value of the objective function for the cases with just uncertain supply or just uncertain demand, as a percentage of the value when both parameters are uncertain. We used the results of the instances with three time periods, one vehicle and 200 scenarios and show the average value over the five instances with the indicated size, separated by the different number of customers, from 5 to 25



(a) Objective function and penalty cost



(b) Number of routes and visits



(c) Relative VSS and EVPI

Figure 3: Results with either stochastic supply or stochastic demands or both

customers (we omitted the results with 30 customers since one of the instances was not solved to optimality within the time limit). The aim of this experiment was to analyze the pooling effect of the customer demands in relation to the number of customers considered. In the figure, it can be observed that the values for the case with just demand uncertainty tend to decrease as we increase the number of customers, getting closer to the case in which all the parameters are deterministic. This may be explained by the pooling effect as it is more likely that the uncertain demands compensate each other when we consider a larger number of customers. On the other hand, the values for the case with just uncertain supply tend to increase when the number of customers grows. This may be explained by the fact that when more customers are considered then more lost demand is to be expected given the uncertain nature of the supply.

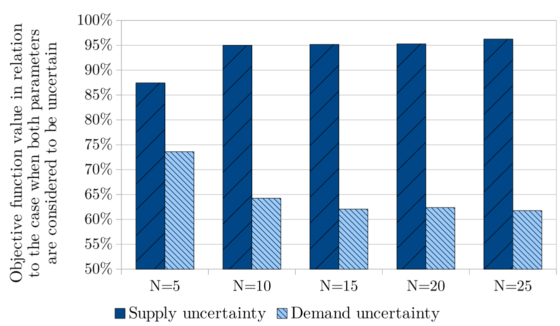
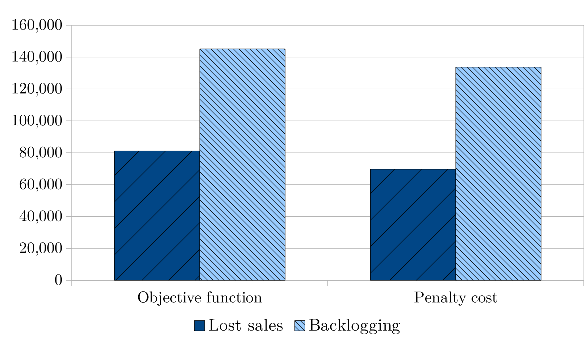


Figure 4: Results with either stochastic supply or stochastic demands for different numbers of customers

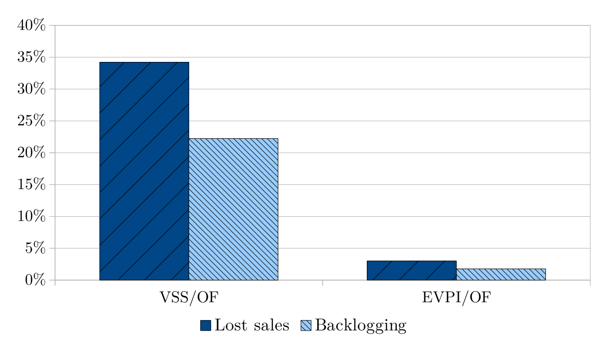
5.3. *Results with the backlogging formulation*

This section shows the results with the formulation using backlogging as the recourse in case of out-of-stock situations, as presented in Section 3.2. The objective is to show the differences between the solutions with this recourse and the ones with lost sales as the recourse. For this, we set an uncertainty level of 60% for both supply and demand (i.e., $\epsilon^r = \epsilon^d = 0.6$) and imposed a time limit of two hours to solve each instance with both formulations. We used the same supply and demand realizations for both formulations and we used 100, 200 and 500 scenarios. We also used the same values for the backlogging and lost sales costs (a_i). For comparison purposes, we only analyze the results for those instances solved to optimality within the time limit with both formulations for the given uncertainty level (329 instances). For all the solutions in both cases, Figure 5a shows the average objective function and penalty cost (recourse) values, while Figure 5b displays the VSS and EVPI w.r.t. the objective function value. As expected, since in the backlogging case the unserved demands accumulate from one period to the other, the average backlogged amounts are considerably higher than the average lost sales (in the lost sales case). This results in larger penalty values and, consequently, in larger values of the objective function (Figure 5a). Also, notice that the relative VSS and EVPI are smaller for the backlogging case than for the lost sales case (Figure 5b), which may be explained by the fact that in the backlogging case there are larger values of the objective function. However, for the backlogging case, the average VSS still represents more than 20% of the objective function value, which also justifies the use of the stochastic programming model. Similarly to the lost sales case, the EVPI is relatively small when compared to the objective function value (Figure 5b). This may imply that even if the supplier had the perfect supply and demand forecast, this would not result in significantly better solutions, showing the robustness of the solutions obtained with the stochastic model.

Similarly to the lost sales case, when backlogging is used as recourse in the second stage, we could also observe the protection mechanism consisting of performing more visits with smaller delivery quantities for increasing values of the uncertainty levels (for both ϵ^r and ϵ^d). For the sake of brevity we do not display these results. It is also worth mentioning that the average CPU time to solve the considered instances was 467 seconds with the backlogging formulation, compared to 811 seconds with the lost sales case recourse. It is also worth mentioning that the average VSS obtained for the lost sales case in this experiment is larger than that observed in the previous experiment and shown in Table 1. This is a result of the subset of instances that are used for comparison purposes, since in each experiment we consider the results for those instances solved to optimality within the time limit for the analyzed cases.



(a) Objective function and penalty cost



(b) Relative VSS and EVPI

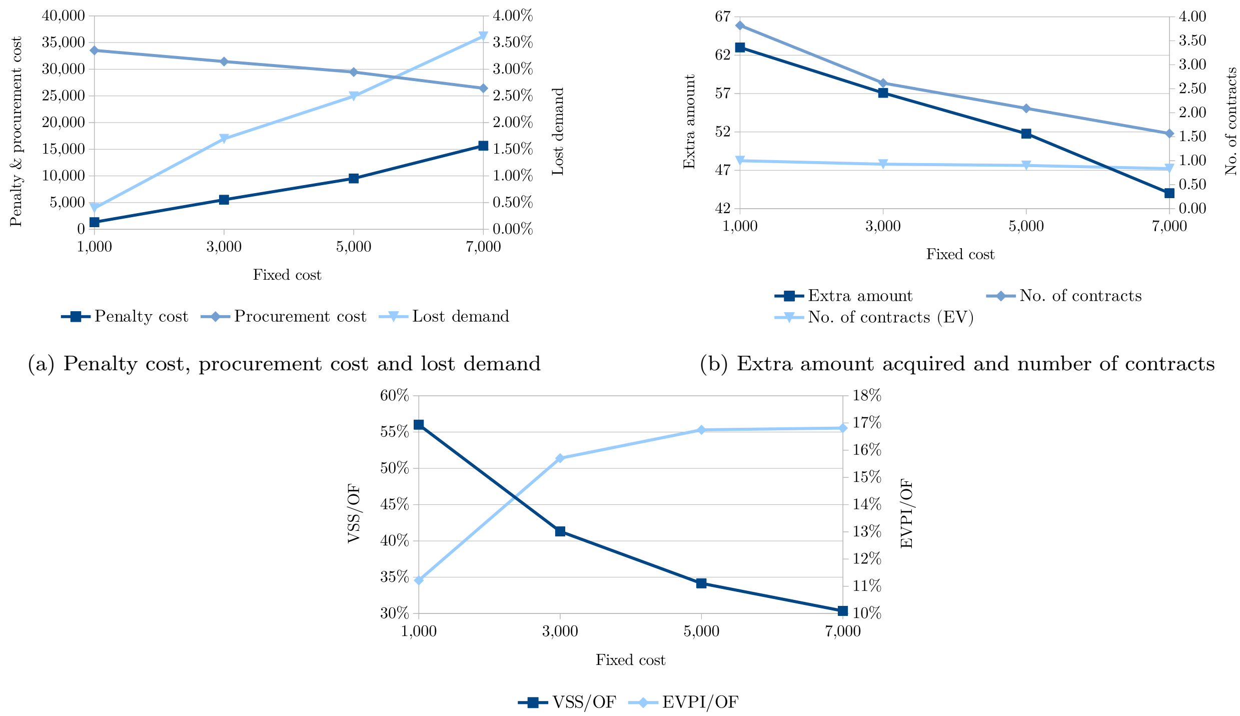
Figure 5: Comparison between the backlogging and lost sales formulation results

5.4. Results with the CRC and lost sales formulation

This section shows the results for the CRC formulation with lost sales, presented in Section 3.3. We analyze the results for increasing values of the fixed contracting cost, namely $f \in \{1{,}000;\ 3{,}000;\ 5{,}000;\ 7{,}000\}$. For this, we set an uncertainty level of 60% for both supply and demand (i.e., $\epsilon^r = \epsilon^d = 0.6$) and imposed a time limit of two hours to solve each instance. We used the same supply and demand realizations for each value of the fixed cost and, as in the previous experiments, we used 100, 200 and 500 scenarios. For comparison purposes, we only analyze the results for those instances solved to optimality within the time limit for all the four values used for the fixed cost (317 instances). Figure 6a shows the average values of the penalty (lost sales penalty) and procurement cost as well as the average lost demand of the solutions; while Figure 6b displays the average extra amount acquired in each period and number of contracts; and Figure 6c indicates the VSS and EVPI w.r.t. the objective function value. Additionally, we show in Figure 6b the average number of contracts in the EV solutions, computed as part of the VSS.

The results show that as we increase the values of the fixed contracting cost, fewer contracts are reserved by the supplier (thus, less extra amount is acquired) (Figure 6b), which results in considerably larger values of the lost demand (and then in larger penalties) (Figure 6a). This observation highlights the trade-off between the cost of the lost demand, and the procurement and delivery cost of the extra amounts. Figure 6c shows that increasing values of the fixed contracting cost result also in larger relative EVPI, as a consequence of the increasing total cost of the solution of the stochastic model. On the other hand, the relative VSS decreases when we increase f. This latter observation implies that a large value for the fixed contracting cost does not compensate the penalties incurred by lost sales, thus making the model less reactive against uncertainties, which reduces the gains of the stochastic programming formulation. Also, in Figure 6b shows that the solutions using an expected value for the stochastic parameters (EV solution) tend to underestimate the number of contracts required, which leads to large lost demand values. The last observation highlights the advantages of the stochastic programming model and the potential consequences of ignoring the stochastic nature of the parameters in the problem.

We also analyzed the results with this formulation for increasing values of the uncertainty levels ϵ^r and ϵ^d, from 0.2 to 0.8 (increasing both supply and demand uncertainty by 0.2), for a value of the fixed contracting cost (f) of 5,000. We imposed a maximum running time of two hours to solve each instance with the formulation. We solved all the formulations with 100, 200 and 500 scenarios, but only show the results for those instances solved to optimality within the time limit for the four cases of the uncertainty levels. Figure 7a displays the average values of the penalty (lost sales penalty), procurement and contracting cost; while Figure 7b shows the average total number of visits (No. of visits), delivery size (Avg. q) and extra amount acquired in each period (Avg. w); Figure 7c shows the average number of contracts in the stochastic and EV solutions (computed as part of the VSS); and Figure 7d indicates the average VSS and EVPI w.r.t. the objective function value. In the charts we can observe that increasing uncertainty levels resulted in larger values of the penalty costs as a consequence of the larger values of the lost demand. This also results in increasing values for the procurement and contracting costs (Figure 7a). In this case it is also possible to observe the protection mechanism consisting of planning more visits (and now more contracts) in the first stage and performing smaller deliveries in the second stage (Figure 7b). Increasing uncertainty levels resulted also in larger relative VSS and EVPI (Figure 7d). It also possible to observe in Figure 7c that the number of contracts in the EV solutions remains relatively stable for increasing uncertainty levels



(a) Penalty cost, procurement cost and lost demand

(b) Extra amount acquired and number of contracts

(c) Relative VSS and EVPI

Figure 6: Results with the CRC and lost sales formulation for different contracting costs

while the stochastic solutions adapt to this increase. This observation may partially explain the increasing VSS (Figure 7d).

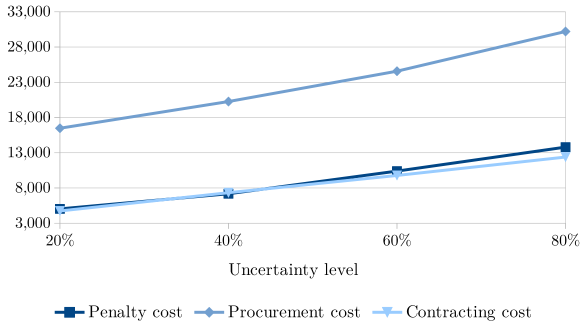
Figure 8 shows the average penalty, procurement and contracting cost for all the instances that were optimally solved within the time limit when considering uncertainty in both supply and demand, only in supply, or only in demand. Analogously to the experiments with the lost sales formulation (Section 5.2), it can be observed that smaller values of the penalty cost are obtained when we consider only uncertain demand compared to the case with only uncertain supply. This may again be due to the pooling effect of the customer demands within each time period. As a result, less procurement and contracting costs are incurred in the case with only demand uncertainty when compared to the other two cases.

5.4.1. Setting a service level in the CRC formulation

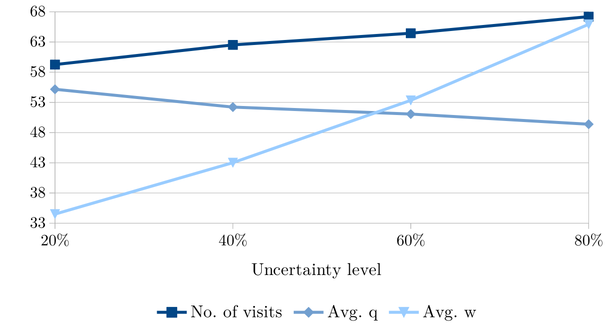
In this section we show the results with the CRC and lost sales formulation when we impose a minimum service level. For this, we included constraints of the form

$$\sum_{i \in \mathcal{C}} \sum_{t \in \mathcal{T}} u_i^t \leq (1-\kappa) \sum_{i \in \mathcal{C}} \sum_{t \in \mathcal{T}} d_{is}^t \qquad \forall s \in \mathcal{S} \tag{53}$$

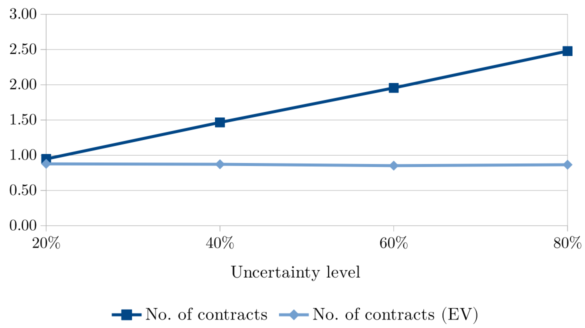
to the formulation. These constraints limit the total lost demand in each scenario to a fraction of the scenario total demand, set by the required service level κ. This parameter defines the minimum percentage of the total demand that must be served in each scenario. We considered five cases, four with different service levels (90%, 94%, 97% and 100%) and the base case ($\kappa = 0$). Again, we set an uncertainty level of 60% for both supply and demand (i.e., $\epsilon^r = \epsilon^d = 0.6$) and imposed a time limit of two hours to solve each instance. We compare and analyze the results for those instances solved to optimality within the time



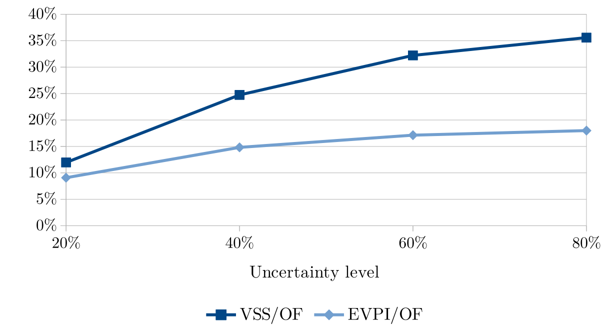
(a) Penalty, procurement and contracting costs



(b) Number of visits, delivery size and extra amount acquired



(c) Number of contracts in the stochastic and EV solutions



(d) Relative VSS and EVPI

Figure 7: Results with the CRC and lost sales formulation for different uncertainty levels

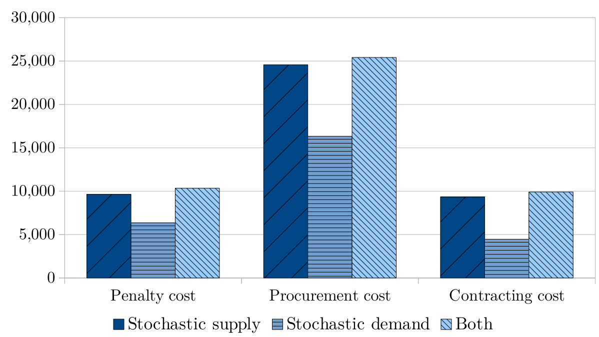
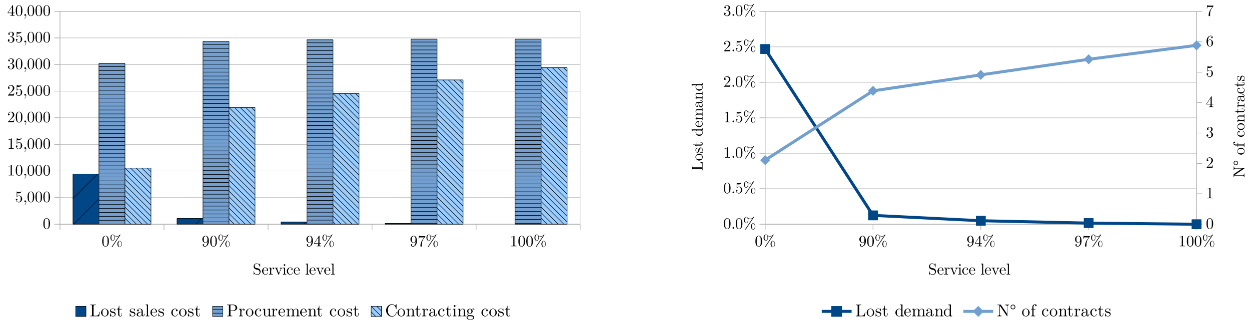


Figure 8: Results with the CRC and lost sales formulation with either stochastic supply or demand or both

limit for all the five values of κ (314 instances). Figure 9a shows the average lost sales, procurement and contracting costs of all the considered instances for the respective service level (given in the horizontal axis); and Figure 9b shows the average percentage of the demand that is lost and the number of contracts in each case.

As expected, increasing values of the minimum service level (κ) resulted in reduced values for the lost sales cost (and lost demands). This reduction in lost demand is achieved by increasing the number of contracts and the extra amount acquired from the external source, which increases significantly the contracting and procurement costs, respectively. It is worth pointing out that from a value of 90% and higher for the service level, the values of the lost demand decreased considerably (0.1% of lost demand, on average) compared to the case without service level. It is also worth mentioning that there are several different ways of modeling service level requirements for production and distribution systems. For more details we refer to Tempelmeier (2013) and Gruson et al. (2018).



(a) Lost sales, procurement and contracting costs

(b) Lost demand and number of contracts

Figure 9: Results with the CRC and lost sales formulation with service level constraints

5.5. *Results with the progressive hedging-based heuristic*

In this section we analyze the performance of the PH-based heuristic approach. We compare its performance with the results obtained with CPLEX with a time limit of two hours. We used three stopping criteria for the heuristic: *i*) a time limit of 600 seconds; *ii*) a maximum of S (number of scenarios) iterations; and *iii*) a maximum of 20 iterations without improvement. We set the values of the parameters of the heuristic to $\varepsilon^y = \varepsilon^x = 0.2$, $M^y = M^x = 0.8$ and $b_0 = 1.0$. For the CRC case we set $\varepsilon^v = 0.5$, $M^v = 1.0$. All these values were determined through preliminary experiments. In addition, the ILS-based hybrid method, used to solve the scenario subproblems and in the feasible solution generation procedure, was configured as in Alvarez et al. (2020, 2018), using as stopping criteria the running time limit of 0.5 and five seconds in the former and latter case, respectively, and the maximum number of iterations, which was set to 1,000.

For the tables in this section, columns labeled with 'CPLEX' show the results of the solver, where '#F' and '#O' show the number of feasible and optimal solutions found by the solver within the time limit, 'Opt gap' shows the optimality gap of the solutions found by the solver and 'Time' shows the running time of the solver (in seconds). Columns labeled with 'PH' show the results with the PH-based heuristic. 'Dif UB' shows the relative difference between the value of the best solutions found by the heuristic (z^h) and the value of the best feasible solutions found by the solver (z^f), computed using the formula $100 \times (z^h - z^f)/z^f$. Column 'Dif LB' shows the optimality gap of the obtained solutions, computed as $100 \times (z^h - \underline{z})/\underline{z}$, where $\underline{z}$ is the best LB computed by the solver. Finally, 'Time' shows the CPU required by the heuristic (in seconds). The values of 'Dif UB' and 'Dif LB' are computed only over those instances for which the solver could find a feasible solution within the two-hour time limit.

First, in Table 2, we report the results for the lost sales case, setting a value of 60% for the uncertainty levels for both supply and demand ($\epsilon^r = \epsilon^d = 0.6$). Each line shows the average for all the instances with the dimensions shown in columns 1 to 3. The last row (Avg.) displays shows the mean values of the corresponding column. The results show that on average the heuristic is able to find reasonably good feasible solutions within relatively short CPU times when compared to CPLEX. Specifically, the heuristic can find solutions with an average relative difference of less than 1% compared to those provided by the solver. These solutions have an average optimality gap of 2.34% and were obtained in an average CPU time of 619 seconds, which represents 27% of the time spent by the solver. It is worth highlighting that the heuristic was able to find feasible solutions for all the 540 instances considered in this experiment while the solver could find feasible solutions for only 80% of the instances (430 instances) within the time limit and with an average CPU time of more than 2,200 seconds.

Instance set			CPLEX				PH		
					Opt	Time	Dif	Dif	Time
T	K	S	#F	#O	gap (%)	(secs)	UB (%)	LB (%)	(secs)
3	1	100	30	30	0.00	7.77	0.31	0.31	528.92
6	1	100	30	30	0.00	172.83	0.92	0.92	556.27
9	1	100	30	29	0.00	647.21	1.57	1.57	570.75
3	2	100	30	26	0.02	1,822.79	2.01	2.03	550.08
6	2	100	28	10	10.19	5,376.14	-5.75	4.93	565.80
9	2	100	17	5	0.93	6,177.60	5.31	6.30	577.50
3	1	200	30	30	0.00	19.21	0.28	0.28	564.98
6	1	200	30	30	0.00	443.71	1.00	1.00	596.95
9	1	200	30	28	0.00	1,712.54	1.93	1.93	605.43
3	2	200	30	23	0.12	2,853.08	1.74	1.86	597.91
6	2	200	20	6	1.27	5,680.21	4.08	5.43	604.72
9	2	200	11	1	11.50	7,069.93	-4.44	8.60	627.66
3	1	500	30	30	0.00	87.28	0.28	0.28	685.88
6	1	500	29	27	0.01	1,841.89	2.02	2.03	724.73
9	1	500	16	13	0.15	3,286.60	2.69	2.84	698.05
3	2	500	25	14	2.58	4,219.84	-0.16	2.45	740.32
6	2	500	9	2	6.74	6,616.20	-1.23	6.19	698.36
9	2	500	5	0	17.47	7,200.23	-12.29	6.13	647.76
Avg.			430	334	1.56	2,295.26	0.67	2.34	619.00

Table 2: Results with the PH-based heuristic for the lost sales case with $\epsilon^r = \epsilon^d = 0.6$

Regarding the specific algorithmic components leading to the final solutions found by the PH-based heuristic, it is worth mentioning that 119 of the final solutions were found in the third case of the feasible solution generation procedure of the heuristic (Section 4.5), while the remaining (421 solutions) were found by the ILS-based hybrid heuristic component also included in the same procedure. This result highlights the importance of the feasible solution generation procedure included in our heuristic. The results also reveal the advantages of the heuristic to solve larger instances when compared to the solver. For instance, for sets with nine time periods, two vehicles and 200 and 500 scenarios, the heuristic finds solutions that are on average up to 12% better than those found by the solver, which could only find 16 feasible solutions (out of 60 instances) and prove the optimality of one of them. Notice also that for some sets the total time of the heuristic might be larger than 600 seconds. This is due to the fact that we do not stop the execution of the heuristic during inner iterations (i.e., solving the scenario subproblems) but only after outer iterations. It is worth mentioning that the values of '#F' and '#O' in this table do not match those shown in Table 1 since in the latter table we display the results only for those instances solved to optimality for all the four cases of the uncertainty levels (Section 5.2).

We also analyzed the performance of the heuristic under different uncertainty level values. The results are displayed in Table 3. The results show that the heuristic is able to find reasonably good feasible solutions when compared to the solver for different values of the uncertainty levels. In particular, when we consider relatively low uncertainty levels (e.g., $\epsilon^r = \epsilon^d = 0.2$) then the heuristic finds solutions that are on average better than those found by the solver within the time limit. It is also possible to observe the relatively stable behavior of the heuristic in terms of running times for the different uncertainty levels that were tested. It is worth highlighting that for all the considered cases, the heuristic could find feasible solutions for all the 540 problem instances while the solver could do it for a maximum of 85% of the instances (457 out of 540 for $\epsilon^r = \epsilon^d = 0.8$). These results show the advantages of the heuristic when compared to a general-purpose optimization solver.

Uncertainty level		CPLEX				PH		
ϵ^r	ϵ^d	#F	#O	Opt gap (%)	Time (secs)	Dif UB (%)	Dif LB (%)	Time (secs)
0.2	0.2	348	194	5.54	3,832.04	-2.74	3.22	615.41
0.4	0.4	397	296	3.24	2,653.74	-1.27	2.18	621.48
0.6	0.6	430	334	1.56	2,295.26	0.67	2.34	619.00
0.8	0.8	457	356	0.65	2,078.50	1.92	2.62	620.79
0.6	0.0	404	288	1.67	2,771.02	-0.35	1.38	618.05
0.0	0.6	368	261	3.64	3,009.63	-1.82	2.09	619.75

Table 3: Results with the PH-based heuristic for the lost sales case under different uncertainty levels

Additionally, Table 4 shows the results of the PH-based heuristic for the CRC case. We tested the heuristic under different uncertainty levels and for a value of the fixed contracting cost (f) of 5,000. In this case, the heuristic is also able to find feasible solutions for all the 540 instances of each combination of the uncertainty levels. The results show that the heuristic finds reasonably good feasible solutions for the CRC case using a small fraction of the running time spent by the solver.

Uncertainty level		CPLEX				PH		
ϵ^r	ϵ^d	#F	#O	Opt gap (%)	Time (secs)	Dif UB (%)	Dif LB (%)	Time (secs)
0.2	0.2	375	257	4.85	3,146.52	-2.02	3.34	641.12
0.4	0.4	399	313	2.00	2,338.07	1.07	3.24	641.21
0.6	0.6	417	324	2.05	2,282.46	1.20	3.57	638.25
0.8	0.8	452	358	0.86	2,075.85	3.80	4.74	638.73
0.6	0.0	406	290	2.69	2,800.41	-0.64	2.22	640.63
0.0	0.6	393	289	4.31	2,714.34	2.20	7.28	632.92

Table 4: Results with the PH-based heuristic for the CRC case under different uncertainty levels

From all these experiments it was possible to observe that the solutions provided by the heuristic also present the protection mechanisms against the uncertainties of the optimal solutions, as described in the previous sections. In particular, for the lost sales case, the solutions tend to perform more visits and the delivery sizes tend to decrease when we increase the uncertainty levels. For the CRC case, the protection mechanism also includes increasing the number if contracts in the first stage when the uncertainty levels increase. These results highlight the advantages of the heuristic since, in addition to providing high-quality solutions in a small fraction of the CPU time used by the solver, its solutions are robust in terms of behavior against the problem uncertainties. It is worth mentioning that for the sake of brevity we do not display the results of the heuristic algorithm in their full extent.

6. Conclusions

In this paper, we addressed an inventory routing problem under both stochastic product supply and customer demands. We introduced a two-stage stochastic programming formulation considering recourse mechanisms such as lost sales, backlogging and an additional source for the product in a capacity reservation contract setting. We also proposed a progressive hedging-based heuristic algorithm. We have provided several managerial insights regarding the behavior of the optimal solutions under different configurations for the uncertainty levels and costs of the system. Furthermore, the results with the heuristic algorithm showed that it provides high-quality solutions within reasonable running times for instances with a large number of scenarios. Future research could focus on considering other recourse actions as well as working on a multi-stage setting for the problem.

Acknowledgments

The authors gratefully acknowledge the support of the São Paulo Research Foundation (FAPESP), grants 17/06664-9 and 17/13739-5; the Coordination for the Improvement of Higher Education Personnel (CAPES); and the Chair in Supply Chain Operations Planning at HEC Montréal.

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