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Scalable Multi-Stage Stochastic Optimization for Freight Procurement in Transportation-Inventory Systems

The procurement of freight services is an important element for the supply chain management of a shipper (i.e., a manufacturer or retailer) that sources transportation services from the third-party logistics market. Motivated by a practical freight procurement problem faced by shippers, we provide a holistic approach to designing freight procurement strategies for transportation-inventory systems that captures the interconnections between freight procurement, transportation, and inventory management. In view of the supply and demand uncertainties, we consider the problem in a multi-stage decision process that complies with the revealing process of the uncertain data. In the first stage, freight service contracts are procured for the entire planning horizon, while the delivery quantities and inventory levels are determined in the subsequent stages. We introduce a mixed-integer linear programming model for the multi-stage problem. To handle instances of realistic size, we propose a stochastic dual dynamic programming solution approach which is further enhanced through novel feasibility inequalities, optimality inequalities, and a primal-dual lifting method. We prove the validity of the proposed inequalities and conduct extensive numerical experiments which demonstrate that our approach scales to large-scale instances with up to 10^9 scenarios. Compared to methods commonly adopted to solve similar problems, this joint optimization approach could potentially help reduce the total cost for shippers by 19% to 31% based on our generated instances from real-world data and simulations.

Key words: freight procurement, transportation-inventory management, multi-stage stochastic optimization, stochastic dual dynamic programming

1. Introduction

The distribution and storage of commodities are critical activities for any company that acts as a *shipper* in a supply chain. While some have their own capabilities, many shippers rely on the transportation market (i.e., third-party logistics, 3PL) for distributing commodities. 3PL services play a vital part in global trade and the 3PL market is valued at more than one trillion US dollars (USD) worldwide (Allied Market Research 2019). To date, 80% of Fortune 500 companies and 96% of Fortune 100 companies use some form of 3PL services (Akarca 2018). Moreover, given the current surge in shipping costs, especially in the maritime sector (Page 2021), it is highly desirable for the shippers to leverage a systematic optimization approach to optimize freight procurement.

In the transportation market, shippers procure freight services from *carriers* (i.e., 3PL service providers). The basic elements in freight service procurement are *lanes*, and a lane is an origin-destination pair with transportation demand over a period. Freight service procurement on a lane is an auction process with shippers acting as auctioneers and carriers acting as bidders (Alp et al. 2003, Caplice and Sheffi 2003, Lim et al. 2008). Results in the auction on a lane are *capacity contracts* negotiated between the shipper and carriers who win the auction. A capacity contract specifies the number and schedule of shipments to be performed by the carrier on the lane, the capacity of each shipment, as well as the freight rate payable by the shipper. On top of capacity contracts, freight services for single shipments can also be acquired through standard freight rates. For serving the same lane, the freight rates of capacity contracts are typically lower than the standard freight rates.

Normally, the service period of a capacity contract on a lane ranges from several months to two years (Sheffi 2004, Wu et al. 2021). Considering its long service period, typically, a capacity contract has to be determined by a shipper without fully knowing the transportation demand on the lane. Therefore, when determining the capacity contracts on a lane, the shipper should consider the uncertainty in transportation demand and the necessary adjustments under different demand scenarios.

This paper introduces a joint freight procurement and transportation-inventory management problem (FPTMP) under supply and demand uncertainty faced by a shipper. We consider the problem with a single commodity in a discrete and finite time horizon, where the distribution network consists of multiple suppliers and multiple customers. Inventories can be stored at both the suppliers and the customers but must be kept under the corresponding upper limits. Backlogging is allowed both at the supply and demand sides. Freight services are purchased through capacity contracts that must be negotiated in a contracting phase prior to the planning horizon. In the planning horizon, the carriers, acting under such capacity contracts, distribute the commodity on the lanes in the distribution network. In addition, the shipper can also choose to transport the commodity on any lane at any time through a standard rate.

We consider the situation where the supply and demand information is uncertain and is gradually disclosed to the shipper at different time periods during the planning horizon. We, therefore, consider the resulting stochastic FPTMP (SFPTMP) under uncertainty in a multi-stage process. In the first stage, freight service procurement decisions are made such that the capacity contracts between the shipper and the carriers must be determined. The loading quantity for each shipment in the capacity contracts, the volume to be transported through a standard rate on each lane in each period, and the inventory and backlog levels at the suppliers and customers are decided in the subsequent stages after the supplies and demands in these stages become known. The objective of the problem is to minimize the expected total cost incurred by procuring freight services, distributing the commodity, holding inventories, and backlogging supplies and demands.

1.1. Background

This study is motivated by the transportation and inventory management of iron ore in a large Chinese steel manufacturer. The manufacturer imports iron ore from two loading ports near the mines in Australia and Brazil to two unloading ports near its plants in China (i.e., there are four physical shipping lanes in total). Iron ore is purchased from the suppliers using a hybrid of (i) long-term contracts that provide steady supplies in a period covering five to ten years and (ii) short-term agreements based on real-time fluctuations in the market. We refer to Comtois and Slack (2016) for a review of the global iron ore supply chain.

Inventories of iron ore can be stored at the yards of ore mines and steel plants. The manufacturer buys freight services from bulk shipping companies, mainly through contracts of affreightment (COAs) and voyage charters (VCs). COAs are capacity contracts in bulk shipping. The COAs for shipping iron ore are typically signed or renewed at the beginning of a year and a COA typically covers a service period of a year. Supply and demand information for an entire year is not fully known to the manufacturer when signing the COAs. Nevertheless, historical supply and demand data for estimating the distribution is available. In a COA, the shipping company is required to perform multiple shipments between a pair of loading and unloading ports during the service period. For each shipment in a COA, the shipping company is required to provide a ship with the same (or similar) capacity. The loading times of the shipments in a COA can be specified in a shipment schedule agreed by both parties. In practice, shipping companies typically require shipments in a COA to be fairly evenly spread over the service period.

Unlike COAs, VCs are for single shipments. They are more flexible and can be obtained whenever shipping demands arise. Both COAs and VCs stipulate freight rates payable by the shippers. For shipping cargoes between the same pair of ports, the freight rate in a COA is typically much lower than that in a VC. We refer to Wu et al. (2021) for the conceptual descriptions of COAs and VCs. For concrete examples of COAs and VCs, we refer the reader to the template contracts for bulk cargo COAs (i.e., VOLCOA and GENCOA) and VCs (e.g., GENCON and NUVOY) and the explanatory notes provided by BIMCO (2022), which is the largest international shipping association representing shipowners.

The manufacturer thus faces an SFPTMP for arranging its iron ore transportation. This problem typically involves a one-year planning horizon where COAs are determined at the beginning and shipping decisions are made at the subsequent periods when the relevant demand and supply information becomes known.

1.2. Contributions

Our study makes four main contributions:

1. We study a joint freight procurement, transportation, and inventory control problem under supply and demand uncertainty motivated by a real-world application in a large steel manufacturer. Such a problem commonly arises in the supply chain management of a manufacturer or a retailer.
2. We develop a solution approach based on stochastic dual dynamic programming (SDDP) and propose several computational enhancements: the use of feasibility inequalities to avoid big-M coefficients in the formulations, optimality inequalities formulated based on smaller scenario trees, and a primal-dual lower bound lifting procedure based on stage-wise convergence. The approach scales to instances with up to nine stages and 10 scenarios per stage (i.e., 10^9 scenarios).
3. We prove the validity of the inequalities and conduct extensive experiments using instances generated from an existing benchmark suite as well as real-world application data. The results demonstrate that our approach can well solve instances with realistic scales and that our approach performs robustly under different problem settings.
4. We make the code implementation, instances, and detailed results publicly available as a benchmark suite for the SFPTMP.

1.3. Outline

The remainder of this paper is structured as follows. We review the relevant studies in Section 2. We formally describe the SFPTMP in Section 3 and formulate it as a compact MILP model in Section 4. The SDDP approach for solving the problem is described in Section 5. We explain the enhancement strategies for the SDDP approach in Section 6. Computational experiments are reported in Section 7, followed by conclusions in Section 8. We provide all mathematical proofs of the theorems, propositions, and lemmas presented in this paper in Appendix A.

2. Literature Review

In addition to the aforementioned application in seaborne bulk shipping, similar problems also commonly arise in supply chains that, for example, use third-party trucking and airline services. Despite their importance, to the best of our knowledge, no studies in the literature have to date considered the FPTMP or the SFPTMP. In the FPTMP, decisions of commodity distribution, inventory and backlog management, and freight service procurement are considered in an integrated manner. The most closely related studies in the literature are Bertazzi et al. (2015) and Boujemaa et al. (2022). In the problem considered by Bertazzi et al. (2015), a single commodity is shipped from a single supplier to multiple customers with stochastic demands in a finite discrete-time planning horizon. Transportation between the supplier and the customers is outsourced such that a 3PL carrier performs all deliveries on all lanes (with an unlimited capacity on each lane) in one period at a fixed cost. The authors formulated the problem as a dynamic programming model. In view of the curses of dimensionality, they proposed a matheuristic for solving the model. Our study is different from this one. In particular, in the SFPTMP, the capacities of shipments in capacity contracts have to be decided prior to the shipping phase, which contains multiple decision stages, and the transportation costs are volume-dependent. In addition, we propose an exact solution approach for the problem. Boujemaa et al. (2022) considered a shipment assignment problem on a distribution network in a finite discrete-time planning horizon. The commodities are distributed by a set of (contracted) core carriers under guaranteed capacities and freight rates and a set of spot carriers. It is assumed that both the selection of the core carriers and their service conditions have been decided. Demands are stochastic and are dynamically revealed in the planning horizon. The objective is to formulate the best shipment strategy that minimizes the expected total cost incurred by inventories, backlogs, and shipments. The problem was formulated as a linear programming (LP) model and solved by an SDDP approach. In comparison, in the SFPTMP, transportation procurement decisions (including those based on long-term contracts and those based on standard freight rates) are considered in an integrated manner. This makes the SFPTMP fundamentally more complicated than the problem in Boujemaa et al. (2022).

The SFPTMP is related to the transportation procurement problem (TPP), especially the TPP under uncertainties. The TPP, also referred to as the winner determination problem (WDP) of freight services, was introduced by Caplice and Sheffi (2003) who presented the first mixed-integer linear programming (MILP) formulation for this problem. The TPP was later extended by Caplice et al. (2005), Lim et al. (2006, 2008, 2012), and Hu et al. (2016). In these studies, transportation demand on each lane was assumed to be deterministic and known in advance. The decisions considered were to procure freight services (or select bids) from carriers to match the demands at the lowest cost. Various practical issues were considered in these studies, including the limits on the total shipping volume allocated to a carrier, the minimum shipping volume guarantees for carriers, the constraints on the number of carriers selected, the inclusion of the spot carriers, the avoidance of imbalances in freight cost allocation among carriers, as well as the consideration of transit times. The WDP with uncertain lane demands was considered by Ma et al. (2010), Zhang et al. (2014), and Meng et al. (2015) under the assumption that the distribution of the uncertain parameters is known and by Remli and Rekik (2013) and Zhang et al. (2015) under the assumption that the distribution information is not fully available. Remli et al. (2019) extended the work of Remli and Rekik (2013) by taking into account the uncertainties in carriers' capacities. Lee et al. (2021) studied a TPP in liner services with uncertain demands. In their problem, on top of the freight costs, inventory holding costs were also considered to capture the impacts of liner service schedules.

The SFPTMP is fundamentally different from the TPP. In the SFPTMP, the interdependencies between transportation (including commodity delivery and inventory and backlog management) and freight service procurement in a supply chain are fully considered. However, in the TPP, freight service procurement is determined based on a given transportation plan on each lane in each period. Besides, although uncertainty has been considered in the TPP, to the best of our knowledge, all the relevant papers study the TPP under uncertainty in a two-stage process, either by considering a single-period second stage problem (Ma et al. 2010, Remli and Rekik 2013, Zhang et al. 2014, Meng et al. 2015, Remli et al. 2019) or by assuming the uncertain information in all periods is fully disclosed at the beginning of the second stage (Lee et al. 2021). Unlike these studies, we solve the SFPTMP in a multi-stage process that is more natural and realistic. Finally, it is worth mentioning that while both problems share a similar structure, as most TPPs in the literature are motivated by container transportation, they typically involve a larger number of physical lanes and more side constraints than the SFPTMP. Some TPP studies also assume discrete transportation units (Lim et al. 2006, 2008). In comparison, because it is motivated by bulk shipping, the SFPTMP we consider in this paper involves fewer physical lanes but more decision periods. We also consider continuous transportation units. While the difficulty of solving the TPP mainly lies in its large number of binary variables, the hardness of solving the SFPTMP comes from the multi-stage stochasticity and the intertwined relationships among the decisions in different periods.

Some papers consider transportation procurement as a part of a shipper's operations management problem. Alp et al. (2003) considered a transportation contract design problem that allows a shipper to set parameters of a given contract for the freight service on a single lane with deterministic shipping demands. The objective is to minimize the expected total cost for freight procurement and inventory and backlog management. Bertazzi et al. (2000) considered a shipment problem for a manufacturer where a set of products are shipped between an origin and a destination. The manufacturer should decide the frequencies for shipping each product and the volume of each product in each shipment. With known production rates, the considered objective is to minimize the sum of the inventory cost and the transportation cost per unit time. Stecke and Zhao (2007) investigated an integrated production and transportation planning problem for a make-to-order manufacturing company with a commit-to-delivery business mode. The production date of a product may affect the selection of transportation services for shipping it. The objective is to determine the best production and shipment plan for a set of products. Lu et al. (2017) considered a carrier portfolio problem in which a shipper transports and sells seasonal products to an overseas market where the selling price declines over time. Freight services are procured from carriers that deliver the products with distinct arrival schedules and freight rates. The objective is to determine an optimal strategy for allocating the shipping quantities among the carriers that produces the best expected profit. The SFPTMP is also different from these studies. In particular, in the SFPTMP, freight procurement is a tactical decision that has “lasting effects” on a series of operational decisions while it acts as one of the operational decisions considered in these studies. Moreover, we consider the problem in a multi-stage stochastic process, which further differentiates our research from these studies where shippers make all decisions in a single stage.

Multi-stage stochasticity has been considered by some recent works in the area of transportation research (Bertazzi and Maggioni 2018, Hu et al. 2019, Cavagnini et al. 2022, Boujemaa et al. 2022). However, the corresponding problems are mostly solved through heuristic methods that do not guarantee convergence to optimality (i.e., rolling horizon and progressive hedging approaches). To the best of our knowledge, the only exception is Boujemaa et al. (2022), who, as mentioned above, developed a solution approach based on SDDP.

We solve the SFPTMP through SDDP. The approach was introduced by Pereira and Pinto (1991) and is an extension of the nested Benders or L-shaped decomposition method proposed by Birge (1985). The vast majority of SDDP applications in the literature are in energy operations planning problems, especially the hydrothermal generation scheduling problem (HGSP), which focuses on the management of power generation in a system of hydro and thermal plants to meet energy demand in the face of stochastic water inflows into the hydro-reservoirs (Pereira and Pinto 1991, Shapiro et al. 2013). Similar to the SFPTMP, the HGSP involves a multi-stage decision process. In each stage (typically representing a month), the decision-maker decides, for each reservoir, the outflow through the turbine and that through the spillway. Other applications include portfolio optimization in finance (Valladão et al. 2019) and lot sizing in manufacturing management (Quezada et al. 2022, Thevenin et al. 2022).

It has been shown that SDDP can well solve the HGSP or other multi-stage stochastic optimization problems with “simple” inter-stage linkages. In particular, in the HGSP, stages are connected such that the storage level at the end of each stage provides an initial storage level for the next stage. Meanwhile, in the problem studied by Boujemaa et al. (2022), long-term freight service procurement decisions are given and shipments are "instant" (i.e., completed within one period). Hence, in this problem, any two adjacent decision stages are linked by only the inventory and backlog levels in the previous one.

Compared with the HGSP and other problems typically solved by SDDP, the SFPTMP has two salient features that make it more difficult to solve. First, the decisions in a stage of the SFPTMP may be affected by a series of decisions made in multiple previous stages, including the freight service procured in the first stage, the shipping volumes decided in previous stages (shipments may take multiple periods), and the last-period inventory and backlog levels in the preceding stage. Second, each decision stage in the SFPTMP is still a multi-period decision problem, involving a set of interrelated decisions in shipping management and inventory and backlog control. These features not only complicate the development of the SDDP approach (refer to Section 5), but also make the approach much less effective (refer to Section 7). We thus devised tailored methods for enhancing the performance of the basic SDDP approach (refer to Section 6). To the best of our knowledge, we are the first to develop a highly scalable framework with convergence guarantees for such a challenging problem in transportation-inventory management.

3. Problem Description and Notation

In the SFPTMP, a single commodity is shipped from a set of supply sites (suppliers) $\mathcal{I}^S$ to a set of demand sites (customers) $\mathcal{I}^D$ over a discrete and finite time horizon which consists of a set $\mathcal{T} = \{1, 2, .., \bar{t}\}$ of periods. Let $\mathcal{I} = \mathcal{I}^S \cup \mathcal{I}^D$ be the set of all sites. The amount of the commodity produced or required at site $i \in \mathcal{I}$ in period $t \in \mathcal{T}$ is denoted by $d_{i,t}$, where $d_{i,t} \geq 0$ if $i \in \mathcal{I}^S$ and $d_{i,t} \leq 0$ if $i \in \mathcal{I}^D$. Each site $i \in \mathcal{I}$ can hold an inventory not exceeding the upper bound denoted by $\overline{q}_i$ and we let q_i^0 denote the initial inventory at site $i \in \mathcal{I}$ at the beginning of the planning horizon. Besides, in each period, the leftover supply (i.e., supply that cannot be stocked or shipped out) or unmet demand at site $i \in \mathcal{I}$ can be backlogged.

For modeling simplicity, we assume that the inventory holding cost and the backlog cost at any site in each period are linear functions of the storage volume and of the backlog volume, respectively. In particular, for site $i \in \mathcal{I}$, the unit inventory holding cost in each period is denoted by h_i and the backlogged supply or demand at this site is penalized at the unit price e_i. Considering that leftover supply and unmet demand incur disutilities to suppliers and customers, we assume that $e_i > h_i$, $\forall i \in \mathcal{I}$. Let $\mathcal{L} = \{(i,j) | i \in \mathcal{I}^S, j \in \mathcal{I}^D\}$ be the set of (directed) lanes between the supply and the demand sites. The commodity can be shipped on any lane $(i,j) \in \mathcal{L}$. For simplicity, we assume that it takes $o_{i,j} \in \mathbb{Z}^+$ periods to transport any amount of the commodity on lane $(i,j) \in \mathcal{L}$ in one shipment by any means.

Freight services for shipping the commodity are procured from the transportation market through an auction process. In particular, the shipper first makes inquiries to carriers regarding their services on the lanes in $\mathcal{L}$. The carriers then respond by providing a group of bids on each lane. Capacity contracts (e.g., COAs in sea transportation) for freight services are negotiated based on these bids. Let $\mathcal{B}_{i,j}$ be the set of bids for lane $(i,j) \in \mathcal{L}$. Let also $\mathcal{B} = \bigcup_{(i,j)\in\mathcal{L}} \mathcal{B}_{i,j}$. For any $b \in \mathcal{B}$, we use $i(b) \in \mathcal{I}^S$ and $j(b) \in \mathcal{I}^D$ to represent the supply site and the demand site associated with the bid. Each bid $b \in \mathcal{B}$ contains a set $\mathcal{R}_b$ of shipments. Let $\mathcal{R} = \bigcup_{b \in B} \mathcal{R}_b$. Given any shipment $r \in \mathcal{R}$, we use $t_1(r)$ and $t_2(r)$, where $t_1(r), t_2(r) \in \mathcal{T}$, to denote the periods in which the shipment starts and ends, respectively.

All shipments in one bid $b \in \mathcal{B}$ should have the same capacity, and for modeling simplicity, we assume that it can be any value within the range $[\underline{m}_b, \overline{m}_b] \subseteq \mathbb{R}^+$. The freight rate for purchasing shipment capacities in bid b is denoted by f_b. Because all shipments in one bid share the same capacity, we let $F_b = |\mathcal{R}_b| f_b$ denote the unit price for purchasing capacities for all shipments in bid $b \in \mathcal{B}$. Note that the cost for shipment capacity purchase in a bid is incurred once the relevant capacity contract is settled, and it is independent of the actual shipping volumes. On top of this fixed cost, without loss of generality, we further use g_b to represent the extra cost the shipper has to pay for each unit of the commodity transported through any shipment in bid $b \in \mathcal{B}$. In addition to capacity contracts, the shipper can also transport the commodity through standard freight rates without capacity limits. Let $c_{i,j}$ be the standard freight rate for shipping the commodity on lane $(i,j) \in \mathcal{L}$.

In practice, the supplies and demands, represented by parameters $d_{i,t}$, can be uncertain when the capacity contracts are negotiated and the relevant information is typically disclosed gradually during the planning horizon. To characterize the uncertainty in $d_{i,t}$, we define $\boldsymbol{d}_t = (d_{i,t} | i \in \mathcal{I})$ and further $\boldsymbol{d} = (\boldsymbol{d}_t | t \in \mathcal{T})$. We assume that $\boldsymbol{d}$ evolves as a discrete-time stochastic process with finite support. The process contains a set $\mathcal{P} = \{1, ..., \overline{p}\}$ of *stages*. Each stage $p \in \mathcal{P}$ covers a set $\mathcal{T}_p = \{\underline{t}_p, ..., \bar{t}_p\} \subseteq \mathcal{T}$ of periods. We have $\bigcup_{p\in\mathcal{P}} \mathcal{T}_p = \mathcal{T}$ and $\mathcal{T}_p \cap \mathcal{T}_{p'} = \varnothing$, $\forall p, p' \in \mathcal{P}, p \neq p'$.

At the beginning of each stage $p \in \mathcal{P}$, the shipper observes the realization of the uncertain parameters in $(\boldsymbol{d}_t)_{t=1}^{\bar{t}_p}$. The uncertainty of $\boldsymbol{d}_t$ in a stage $p \in \mathcal{P}$ is captured through a set of possible realizations (i.e., scenarios) of $(\boldsymbol{d}_t)_{t=\underline{t}_p}^{\bar{t}_p}$ which are denoted by Ω_p and indexed by ω. For each stage $p \in \mathcal{P}$, we let $d_{i,t}^{\omega}$ be the demand or supply generated at site $i \in \mathcal{I}$ in period $t \in \mathcal{T}_p$ under scenario $\omega \in \Omega_p$. We further assume that given any stage $p \in \mathcal{P}$, the distribution of scenarios in this stage is independent of the scenarios in other stages, and let ϱ_ω be the probability of scenario $\omega \in \Omega_p$.

The full set of scenarios for all stages in the problem can be represented by a scenario tree, as shown in Figure 1. The path from a node in stage 1 to a node in stage $\overline{p}$, denoted by $\{\omega_1, ..., \omega_{\overline{p}}\}$, where $\omega_p \in \Omega_p$, $\forall p \in \mathcal{P}$, corresponds to a scenario ξ for a realization of the uncertain parameters in $\boldsymbol{d}$. Let Ξ be the set of all such scenarios. For each scenario $\xi \in \Xi$, let $d_{i,t}^{\xi}$ denote the supply or demand at site $i \in \mathcal{I}$ in period $t \in \mathcal{T}$ under this scenario. The probability of scenario $\xi \in \Xi$ is denoted by ρ_ξ. For ease of presentation, in this paper, we will refer to scenarios in Ξ simply as *scenarios* and refer to scenarios ω in any Ω_p as *stage scenarios*. For each scenario $\xi \in \Xi$, we let $\omega_p(\xi) \in \Omega_p$ denote the index of the stage scenario in stage $p \in \mathcal{P}$ associated with this scenario. To impose the non-anticipativity constraints in the SFPTMP, for each $p \in \mathcal{P}$, we introduce the set $\Lambda_p = \{(\xi_1, \xi_2) \in \Xi \times \Xi | \omega_{p'}(\xi_1) = \omega_{p'}(\xi_2), \forall p' = 1, ..., p\}$ which contains all pairs of scenarios $(\xi_1, \xi_2) \in \Xi \times \Xi$ that are indistinguishable in stage p.

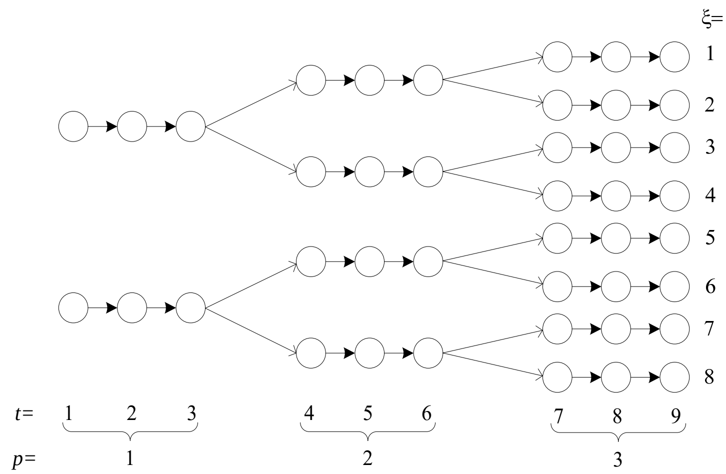

Figure 1 Example of a Scenario Tree.

As illustrated in Figure 2, in the SFPTMP, the shipper can essentially make decisions in $1 + \overline{p}$ stages. The first stage corresponds to the contracting phase prior to the disclosure of any supply-demand information in the planning horizon whereas the $(1+p)$-th decision stage corresponds to stage p in the information disclosure process when the information of supply and demand has been revealed up to this stage, where $p \in \mathcal{P}$. For simplicity, we refer to the first decision stage as stage 0 and let $\mathcal{P}^+ = \{0\} \cup \mathcal{P}$ denote the full set of decision stages.

In stage 0, the shipper signs capacity contracts for freight services on the lanes. As in standard COAs (BIMCO 2022), each contract, which is negotiated based on a bid, specifies the schedule of the shipments, the capacity of each shipment, as well as the associated costs (including the fixed cost for capacity purchase and the freight rate associated with variable shipping volumes) for the freight services on a lane. In the subsequent stages, given the decisions made in the previous stages and the observed supplies and demands in the current stage, the shipper determines how to: (i) allocate the loading quantity for each shipment in the capacity contracts that starts in this stage, (ii) determine the quantity to be transported through the standard freight rate on each lane in each period within this stage, (iii) control the inventory levels at each site in each period in this stage, and (iv) determine the backlogged supply or demand at each site in each period in this stage.

The shipper faces costs from procuring freight services (through capacity contracts or the standard rates), holding inventories, and handling backlogs. The objective is to formulate a joint freight service procurement and transportation-inventory plan that yields the minimum expected total cost.

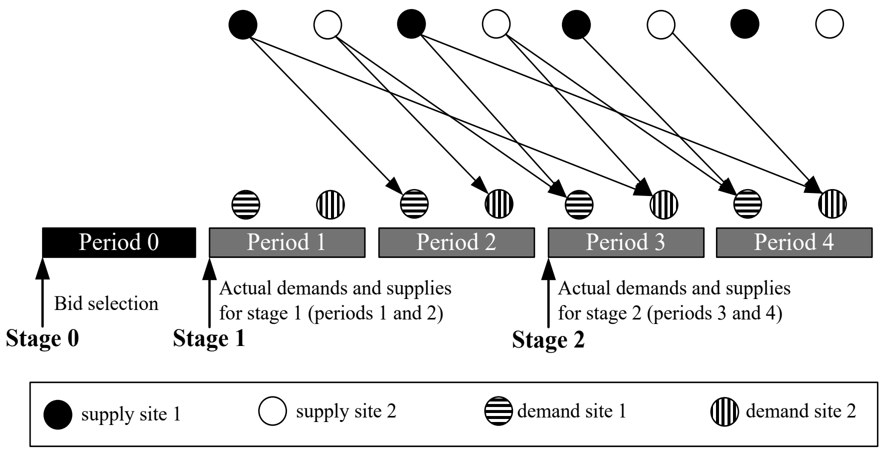

Figure 2 **Illustration of the SFPTMP.**

Finally, the complexity of the SFPTMP is given in the following theorem (see Appendix A.1 for the proof):

THEOREM 1. *The SFPTMP is NP-hard.*

4. Model Formulation

We model the SFPTMP via a space-time network that represents the physical lanes and the shipment schedules simultaneously. Each node in the network is associated with a site $i \in \mathcal{I}$ and a period $t \in \mathcal{T}$, and each arc represents a potential shipment on a lane. Details of the network are explained in Section 4.1 and the model formulated based on this network is given in Section 4.2.

4.1. Space-time Network

To facilitate the formulation of the problem, we construct a space-time network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ and $\mathcal{A}$ represent the set of nodes and the set of arcs, respectively. The node set $\mathcal{N}$ consists of $|\mathcal{T}|$ copies of each site $i \in \mathcal{I}$, i.e., $\mathcal{N} = \{(i,t) | i \in \mathcal{I}, t \in T\}$. Let $\mathcal{N}^S = \{(i,t) \in \mathcal{N} | i \in \mathcal{I}^S\}$ and $\mathcal{N}^D = \{(i,t) \in \mathcal{N} | i \in \mathcal{I}^D\}$ be the sets of nodes associated with supply and demand sites, respectively. The arc set is defined as $\mathcal{A} = \mathcal{A}^1 \cup \mathcal{A}^2$. The arc set $\mathcal{A}^1$ contains arcs that represent shipments in capacity contracts and is defined as $\mathcal{A}^1 = \bigcup_{b \in \mathcal{B}} \mathcal{A}^1_b$, where for each bid $b \in \mathcal{B}$, $\mathcal{A}^1_b = \{((i,t_1),(j,t_2)) | (i,t_1),(j,t_2) \in$

$\mathcal{N}, i = i(b), j = j(b), t_1 = t_1(r), t_2 = t_2(r), r \in \mathcal{R}_b\}$. The arc set $\mathcal{A}^2$ represents the possibilities of transportation via shipments acquired through the standard freight rates and we have $\mathcal{A}^2 = \{((i,t_1),(j,t_2)) | t_2 = t_1 + o_{i,j}, (i,t_1) \in \mathcal{N}^S, (j,t_2) \in \mathcal{N}^D\}$. Finally, corresponding to each stage $p \in \mathcal{P}$, we let $\mathcal{N}_p = \{(i,t) \in \mathcal{N} | t \in \mathcal{T}_p\}$ and $\mathcal{A}_p = \{((i,t_1),(j,t_2)) \in \mathcal{A} | (i,t_1) \in \mathcal{N}_p\}$ represent the sets of nodes and arcs associated with this stage, respectively.

For notational simplicity, we will use n and (i,t) interchangeably to represent a node and a and $((i,t_1),(j,t_2))$ interchangeably to represent an arc. Given any node $n \in \mathcal{N}$, the sets of its outgoing and incoming arcs are written as

$$A^+(n) = \{((i,t_1),(j,t_2)) \in \mathcal{A} | (i,t_1) = n\} \quad \text{and} \quad A^-(n) = \{((i,t_1),(j,t_2)) \in \mathcal{A} | (j,t_2) = n\},$$

respectively.

With slight abuse of notation, we redefine some parameters in the problem to cast them into the network structure. First, for each node $n = (i,1) \in \mathcal{N}$, let $q_n^0 = q_i^0$ be the initial inventory at node n and for each node $n = (i,t) \in \mathcal{N}$, let $\overline{q}_n = \overline{q}_i$ be the upper bound of the inventory that can be stored at this node. Second, we use $d_{\omega,n} = d_{i,t}^{\omega}$ to represent the supply or demand at node $n = (i,t) \in \mathcal{N}_p$ under stage scenario $\omega \in \Omega_p$ in any stage $p \in \mathcal{P}$. We also use $d_{\xi,n} = d_{i,t}^{\xi}$ to represent the supply or demand at node $n = (i,t) \in \mathcal{N}$ under scenario $\xi \in \Xi$. Third, for each node $n = (i,t) \in \mathcal{N}$ let $h_n = h_i$ denote its unit inventory holding cost and let $e_n = e_i$ denote the unit penalty cost associated with the backlogged supply or demand at this node. Finally, for each arc $a \in \mathcal{A}$, we use c_a to represent the unit (variable) transportation cost on this arc. For each $a \in \mathcal{A}$, c_a is set as:

$$c_a = \begin{cases} g_b, & \text{if } a \in \mathcal{A}_b^1, b \in \mathcal{B}, \\ c_{i,j}, & \text{if } a = ((i,t_1),(j,t_2)) \in \mathcal{A}^2. \end{cases}$$

4.2. The Compact Model

We formulate the problem as a multi-stage stochastic optimization model in a compact form. Table 1 lists the decision variables used in the model.

Table 1: Decision Variables in the SFPTMP.

Decision variables in stage 0:	
x_b	binary variable, which equals 1 if and only if the shipper accepts bid $b \in \mathcal{B}$.
y_b	continuous variable, which represents the capacity purchased by the shipper for each shipment in the capacity contract associated with bid $b \in \mathcal{B}$.
Decision variables in stages $p \in \mathcal{P}$:	
$z_{\xi,a}$	continuous variable, which represents the volume of the commodity allocated on arc $a \in \mathcal{A}$ under scenario $\xi \in \Xi$.
$u_{\xi,n}$	continuous variable, which represents the inventory level at node $n \in \mathcal{N}$ under scenario $\xi \in \Xi$.
$v_{\xi,n}$	continuous variable, which represents the volume of the supply or demand backlogged at node $n \in \mathcal{N}$ under scenario $\xi \in \Xi$.

The SFPTMP can be formulated as an MILP model denoted by **P** as follows:

$$\mathbf{P} = \min \sum_{b\in\mathcal{B}} F_b y_b + \sum_{\xi\in\Xi} \rho_\xi \left(\sum_{n\in\mathcal{N}} (h_n u_{\xi,n} + e_n v_{\xi,n}) + \sum_{a\in\mathcal{A}} c_a z_{\xi,a} \right) \tag{1}$$

s.t.

$$y_b \geq \underline{m}_b x_b \qquad \forall b\in\mathcal{B} \tag{2}$$

$$y_b \leq \overline{m}_b x_b \qquad \forall b\in\mathcal{B} \tag{3}$$

$$z_{\xi,a} \leq y_b \qquad \forall a\in\mathcal{A}_b^1, \forall b\in\mathcal{B}, \forall \xi\in\Xi \tag{4}$$

$$u_{\xi,n_1} + v_{\xi,n_1} = d_{\xi,n_1} + u_{\xi,n_2} + v_{\xi,n_2} - \sum_{a\in A^+(n_1)} z_{\xi,a} \qquad \forall n_1 = (i,t), n_2 = (i,t-1) \in \mathcal{N}^S, \forall \xi\in\Xi \tag{5}$$

$$u_{\xi,n} + v_{\xi,n} = d_{\xi,n} + q_n^0 - \sum_{a\in A^+(n)} z_{\xi,a} \qquad \forall n = (i,1) \in \mathcal{N}^S, \forall \xi\in\Xi \tag{6}$$

$$u_{\xi,n_1} - v_{\xi,n_1} = d_{\xi,n_1} + u_{\xi,n_2} - v_{\xi,n_2} + \sum_{a\in A^-(n_1)} z_{\xi,a} \qquad \forall n_1 = (i,t), n_2 = (i,t-1) \in \mathcal{N}^D, \forall \xi\in\Xi \tag{7}$$

$$u_{\xi,n} - v_{\xi,n} = d_{\xi,n} + q_n^0 + \sum_{a\in A^-(n)} z_{\xi,a} \qquad \forall n = (i,1) \in \mathcal{N}^D, \forall \xi\in\Xi \tag{8}$$

$$u_{\xi,n} \leq \overline{q}_n \qquad \forall n\in\mathcal{N}, \forall \xi\in\Xi \tag{9}$$

$$z_{\xi_1,a} = z_{\xi_2,a} \qquad \forall a\in\mathcal{A}_p, \forall (\xi_1,\xi_2)\in\Lambda_p, \forall p\in\mathcal{P} \tag{10}$$

$$x_b \in \{0,1\} \qquad \forall b\in\mathcal{B} \tag{11}$$

$$y_b \geq 0 \qquad \forall b\in\mathcal{B} \tag{12}$$

$$u_{\xi,n} \geq 0 \qquad \forall n\in\mathcal{N}, \forall \xi\in\Xi \tag{13}$$

$$v_{\xi,n} \geq 0 \qquad \forall n\in\mathcal{N}, \forall \xi\in\Xi \tag{14}$$

$$z_{\xi,a} \geq 0 \qquad \forall a\in\mathcal{A}, \forall \xi\in\Xi. \tag{15}$$

The objective function (1) aims to minimize the expected total cost, including the cost of purchasing freight services using capacity contracts, the expected cost of holding inventories and backlogging supplies or demands at the nodes, and the expected cost for sending flows on the arcs. Given a bid, constraints (2) and (3) set the lower bound and upper bound for the capacity of each shipment associated with this bid. Constraints (4) ensure that under any scenario, the actual volumes of the commodity allocated on the arcs associated with shipments in capacity contracts do not exceed their contractual capacities. Constraints (5)–(8) define the relationship among inventory levels and backlogged quantities at the nodes and the flow volumes allocated on the relevant arcs. Constraints (9) require the inventory level at a node to not exceed its upper bound. Constraints (10) are non-anticipativity constraints. These constraints impose the requirement that the flows allocated on an arc in a stage under any two scenarios ξ_1 and ξ_2 that are indistinguishable up to this stage must be identical. Finally, constraints (11)–(15) define the domains for the decision variables.

It is mentionable that in $\mathbf{P}$, the non-anticipativity requirements are imposed only on the flow variables $z_{\xi,a}$, through constraints (10). The following proposition shows that constraints (10) are sufficient to impose non-anticipativity requirements for the SFPTMP.

PROPOSITION 1. *Problem* $\mathbf{P}$ *satisfies the non-anticipativity requirements in the SFPTMP.*

With a given scenario tree, the above formulation forms a deterministic optimization problem. However, the scale of the problem grows exponentially with the number of stages ($|\mathcal{P}|$) and the number of scenarios in each stage ($|\Omega_p|$, $p \in \mathcal{P}$). As a result, only very small instances of the SFPTMP can be solved by applying a general-purpose optimization solver directly on the MILP model for $\mathbf{P}$ (refer to Section 7.4.1). In the next section, we propose an SDDP approach for solving instances of large scale.

5. A Stochastic Dual Dynamic Programming Approach

By exploiting the stage-wise independence of the scenario tree, SDDP decomposes the original problem into stage-wise problems, approximates the cost of the problems in the subsequent stages through Benders cuts, and converges in a finite number of steps to an optimal solution. For the analyses of the statistical properties and convergence of the SDDP approach, we refer to Shapiro (2011). The framework of the SDDP approach for solving the SFPTMP is presented by Algorithm B.1 in Appendix B. The approach follows an iterative procedure and each iteration l in the approach consists of a *sampling* step, a *forward* step, and a *backward* step. In the sampling step, we select a subset Ξ^l of scenarios from Ξ.

In the forward step, we solve problem $\mathbf{P}$ under each sampled scenario $\xi \in \Xi^l$. Given a scenario, $\mathbf{P}$ is decomposed into a set of problems, each corresponding to a stage $p \in \mathcal{P}^+$. A problem in stage $p \in \mathcal{P}^+ \setminus \{\overline{p}\}$ is characterized by an *expected cost-to-go function* (or *cost-to-go function* for short) that estimates the lower bound of the expected total cost incurred in the subsequent stages. Because the probability distribution of the scenarios satisfies stage-wise independence, problems under different scenarios in the same stage $p \in \mathcal{P}^+ \setminus \{\overline{p}\}$ share the same cost-to-go function. Let Ψ_p denote the cost-to-go function for problems in stage p. On top of the cost-to-go function, given any scenario $\xi \in \Xi^l$, the forward-step problem in any stage $p \in \mathcal{P}$ is also characterized by a set of state variables $\bar{\boldsymbol{\chi}}_{\xi,p-1}$ obtained by solving the associated problem in the previous stage, which is referred to as the *parent problem* of this problem.

In the forward step of any iteration, we have one problem in stage 0, which is denoted by $\mathbf{P}_0(\Psi_0)$. The forward step starts by solving this problem. Then, corresponding to each sampled scenario $\xi \in \Xi^l$, we solve problems denoted by $\mathbf{P}_{\xi,p}(\bar{\boldsymbol{\chi}}_{\xi,p-1}, \Psi_p)$ from stage $p=1$ to stage $p=\overline{p}$.

In the backward step, corresponding to each scenario $\xi \in \Xi^l$, a subproblem denoted by $\mathbf{Q}_{\xi,\omega,p}(\bar{\boldsymbol{\chi}}_{\xi,p-1}, \Psi_p)$ is created for each stage scenario $\omega \in \Omega_p$ in each stage $p \in \mathcal{P}$. We solve the dual of these problems to generate valid cuts for updating the cost-to-go functions.

Finally, the SDDP approach terminates when the lower bound is stable or when a preset time limit is reached. In the following subsections, we explain the details of each step.

5.1. The Sampling Step

In the sampling step of iteration l, a subset of scenarios (denoted by Ξ^l) is sampled from the original set Ξ. We have $|\Xi^l| \leq |\Xi|$. The scenarios in Ξ^l are sampled randomly from the original scenario set Ξ based on the distribution $\{\rho_\xi : \xi \in \Xi\}$.

5.2. The Forward Step

In the forward step of iteration l, the problem $\mathbf{P}$ is solved under each scenario $\xi \in \Xi^l$, and given a scenario ξ, the problem is decomposed into $|\mathcal{P}^+|$ subproblems, each corresponding to a decision stage. We let $\mathbf{P}_0$ denote the first-stage problem, and let $\mathbf{P}_{\xi,p}$ denote the problems in stages $p \in \mathcal{P}$ under each scenario $\xi \in \Xi^l$.

As a problem in a stage is partially defined by the decisions made in the previous stages, the communication between problems in different stages must be carefully established. Also, because of the stage-wise solution procedure, such inter-stage communication only exists between problems in adjacent stages. In particular, given a scenario $\xi \in \Xi^l$, the problem $\mathbf{P}_{\xi,p}$ in any stage $p \in \mathcal{P}$, is characterized by the solution of its parent problem i.e., problem $\mathbf{P}_{\xi,p-1}$ or $\mathbf{P}_0$. In the following, we explain the state variables obtained by solving a problem in the forward step that are used to define the problem in the next stage.

To begin with, given a scenario $\xi \in \Xi^l$, the capacities of shipments in the capacity contracts parameterized in problem $\mathbf{P}_{\xi,p}$ in any stage $p \in \mathcal{P}$ should remain consistent with those determined in the parent problem in stage $p-1$. We let $\bar{\boldsymbol{y}}_{\xi,p-1} = (\bar{y}_{\xi,p-1,b} | b \in \mathcal{B})$ be the vector of such capacities obtained by solving the parent problem.

In addition, suppose $p \geq 2$, then the problem $\mathbf{P}_{\xi,p}$ (for any $\xi \in \Xi^l$) is also subject to the inventory levels, backlogged supplies or demands of certain nodes, and the flows of certain arcs that are determined in its parent problem. In order to characterize this information, we define the following sets of nodes and arcs that are critical for transferring information between stages. First, for the nodes, we let $\widetilde{\mathcal{N}}_p = \{n | n = (i, \bar{t}_p) \in \mathcal{N}_p\}$ denote the set of nodes whose inventory levels and backlogged supplies or demands are passed on to the next stage, $\forall p \in \mathcal{P} \setminus \{\bar{p}\}$. Especially, for stage 0, we define $\widetilde{\mathcal{N}}_0 = \{(i, 0) | i \in \mathcal{I}\}$. Second, as for the arcs, we use $\overrightarrow{\mathcal{A}}_{(p_1,p_2)} = \{a | a = (n_1, n_2) \in \mathcal{A}, n_1 \in \mathcal{N}_{p_1}, n_2 \in \mathcal{N}_{p_2}\}$ to represent the set of arcs directed from nodes in stage p_1 to nodes in stage p_2, where $p_1, p_2 \in \mathcal{P}$ and $p_2 > p_1$. Further, $\forall p \in \mathcal{P}$, let $\widetilde{\mathcal{A}}_p = \bigcup_{p_1=1}^{p} \bigcup_{p_2=p+1}^{\overline{p}} \overrightarrow{\mathcal{A}}_{(p_1,p_2)}$ be set of arcs that link nodes $n \in \bigcup_{p_1=1}^{p} \mathcal{N}_{p_1}$ with nodes $n \in \bigcup_{p_2=p+1}^{\overline{p}} \mathcal{N}_{p_2}$. Also, we especially have $\widetilde{\mathcal{A}}_0 = \varnothing$ and $\widetilde{\mathcal{A}}_{\overline{p}} = \varnothing$.

Based on these sets of nodes and arcs, for problem $\mathbf{P}_{\xi,p}$ (where $p \in \mathcal{P}$ and $\xi \in \Xi^l$), we define the following state variables that are determined in its parent problem (i.e., $\mathbf{P}_{\xi,p-1}$ or $\mathbf{P}_0$) and affect this problem. First, for each node $n \in \widetilde{\mathcal{N}}_{p-1}$, we let $\bar{u}_{\xi,p-1,n}$ and $\bar{v}_{\xi,p-1,n}$ denote the inventory level and the backlogged quantity at node n that are determined by solving the parent problem in stage $p-1$. Especially, for $p-1=0$, we define $\bar{u}_{\xi,0,n} = q_i^0$ and $\bar{v}_{\xi,0,n} = 0$, $\forall n = (i,0) \in \widetilde{\mathcal{N}}_0$. Besides, for each arc $a \in \widetilde{\mathcal{A}}_{p-1}$, we use $\bar{z}_{\xi,p-1,a}$ to represent the flow on this arc determined by solving the parent problem. Let $\bar{\boldsymbol{u}}_{\xi,p} = (\bar{u}_{\xi,p,n} | n \in \widetilde{\mathcal{N}}_p)$, $\bar{\boldsymbol{v}}_{\xi,p} = (\bar{v}_{\xi,p,n} | n \in \widetilde{\mathcal{N}}_p)$, and $\bar{\boldsymbol{z}}_{\xi,p} = (\bar{z}_{\xi,p,a} | a \in \widetilde{\mathcal{A}}_p)$. To further simplify the notation, we define $\bar{\boldsymbol{\chi}}_{\xi,p} = ((\bar{\boldsymbol{y}}_{\xi,p})^\top, (\bar{\boldsymbol{u}}_{\xi,p})^\top, (\bar{\boldsymbol{v}}_{\xi,p})^\top, (\bar{\boldsymbol{z}}_{\xi,p})^\top)^\top$.

In the following, given a scenario $\xi \in \Xi^l$ in iteration l of the SDDP approach, we first describe the formulation of problem $\mathbf{P}_0$ in the first stage (biding stage) and then explain the formulation of problems $\mathbf{P}_{\xi,p}$, $\forall p \in \mathcal{P}$ in the shipping stages.

5.2.1. The Problem in the Bidding Stage Decisions made in the problem in the bidding stage (i.e., $\mathbf{P}_0$) include the selection of bids and the purchase of capacities for the shipments associated with the bids. The associated decision variables are given in the vectors $\boldsymbol{x} = (x_b | b \in \mathcal{B})$ and $\boldsymbol{y} = (y_b | b \in \mathcal{B})$. The problem is also formulated based on a cost-to-go function, denoted by $\Psi_0(\boldsymbol{y})$, which is defined as follows:

$$\Psi_0(\boldsymbol{y}) = \min\{\eta_0 :$$

$$\eta_0 \geq 0, \tag{16}$$

$$\eta_0 \geq (\mu_0^k + (\boldsymbol{\nu}_0^k)^\top \boldsymbol{y}), \ \forall k \in \mathcal{K}_0\}, \tag{17}$$

where $\mu_0^k \in \mathbb{R}$ and $\boldsymbol{\nu}_0^k \in \mathbb{R}^{|\mathcal{B}|}$ are parameters.

Problem $\mathbf{P}_0$ can be formulated as the following MILP model:

$$\mathbf{P}_0(\Psi_0) = \min_{\boldsymbol{x},\boldsymbol{y},\eta_0} \sum_{b \in \mathcal{B}} F_b y_b + \eta_0 \tag{18}$$

$$\text{s.t.} \ \ (2), (3), (11), (12), (16), (17).$$

Objective function (18) minimizes the sum of the cost of capacity purchase and the value of the cost-to-go function.

Let $\boldsymbol{y}^*$ be the vector of optimal solution values of the y variables obtained by solving the above model. We then obtain $\bar{\boldsymbol{\chi}}_{\xi,0}$ for characterizing the problems in stage 1 by letting $\bar{y}_{\xi,0,b} = y_b^*$, $\forall b \in \mathcal{B}, \forall \xi \in \Xi^l$.

5.2.2. Problems in the Shipping Stages In stage $p \in \mathcal{P}$ of the shipping stages, we solve a problem denoted by $\mathbf{P}_{\xi,p}$ under scenario $\xi \in \Xi^l$ in the forward step of iteration l. The problem aims at determining the inventory levels and the backlogging strategies for the nodes in the set $\mathcal{N}_p$ and allocating flows on the arcs in the set $\mathcal{A}_p$. Decision variables used for formulating the problem can be partitioned into three groups.

The first group consists of variables in the following vectors: $\boldsymbol{u}_{\xi,p} = (u_{\xi,n} | n \in \mathcal{N}_p)$, $\boldsymbol{v}_{\xi,p} = (v_{\xi,n} | n \in \mathcal{N}_p)$, and $\boldsymbol{z}_{\xi,p} = (z_{\xi,a} | a \in \mathcal{A}_p)$. These variables control inventory levels, determine backlogged quantities, and allocate flows for the nodes and arcs in $\mathcal{N}_p$ and $\mathcal{A}_p$, respectively.

Variables in the second group are auxiliary variables that make local copies of the variables determined in the parent problem of $\mathbf{P}_{\xi,p}$. In particular, we use the set of variables in vector $\boldsymbol{y}'_{\xi,p} = (y'_{\xi,p,b} | b \in \mathcal{B})$ to represent the "copied" capacity of each shipment in the bids. Besides, for the nodes $n \in \widetilde{\mathcal{N}}_{p-1}$, we use variables in $\boldsymbol{u}'_{\xi,p} = (u'_{\xi,p,n} | n \in \widetilde{\mathcal{N}}_{p-1})$ and $\boldsymbol{v}'_{\xi,p} = (v'_{\xi,p,n} | n \in \widetilde{\mathcal{N}}_{p-1})$, respectively, as the local copies of the inventory and backlogging decisions determined in the parent problem of $\mathbf{P}_{\xi,p}$. Finally, for each arc $a \in \widetilde{\mathcal{A}}_{p-1}$, we introduce the variable $z'_{\xi,p,a}$ to copy the flow allocated on it and let $\boldsymbol{z}'_{\xi,p} = (z'_{\xi,p,a} | a \in \widetilde{\mathcal{A}}_{p-1})$.

The decision variables in the third group are used for ensuring the feasibility of problem $\mathbf{P}_{\xi,p}$. In particular, given any two stages p_1 and p_2 such that $p_1, p_2 \in \mathcal{P}$, $p_1 < p$, and $p_2 = p$, the flows on the arcs in the set $\overrightarrow{\mathcal{A}}_{(p_1,p_2)}$ are determined in the problem in stage p_1 without explicitly considering the inventory restrictions for the corresponding head nodes in $\mathcal{N}_p$ in stage p. This is insufficient to ensure feasibility of problem $\mathbf{P}_{\xi,p}$ as such flows can lead to inventories at certain nodes in $\mathcal{N}_p$ exceeding their upper bounds. To avoid such infeasibility issues, we introduce decision variables $w_{\xi,n}$ for the nodes $n \in \mathcal{N}_p \cap \mathcal{N}^D$ which represent the amounts of overflowed inventories (i.e., inventories beyond $\overline{q}_n$) at these nodes. Let $\boldsymbol{w}_{\xi,p} = (w_{\xi,n} | n \in \mathcal{N}_p \cap \mathcal{N}^D)$.

Finally, for notational simplicity, we use $\boldsymbol{X}_{\xi,p}$ to represent the vector that includes all the decision variables from these three groups in problem $\mathbf{P}_{\xi,p}$.

Problem $\mathbf{P}_{\xi,p}$ is also characterized by a cost-to-go function denoted by $\Psi_p(\boldsymbol{X}_{\xi,p})$ which is defined as follows:

$$\Psi_p(\boldsymbol{X}_{\xi,p}) = \min\{\eta_{\xi,p} :$$

$$\eta_{\xi,p} \geq 0, \tag{19}$$

$$\eta_{\xi,p} \geq \mu_p^k + (\boldsymbol{\nu}_p^k)^\top \boldsymbol{X}_{\xi,p}, \forall k \in \mathcal{K}_p\}, \tag{20}$$

where $\mu_p^k \in \mathbb{R}$ and $\boldsymbol{\nu}_p^k \in \mathbb{R}^N$ are parameters with $N = |\mathcal{B}| + 2|\mathcal{N}_p| + |\mathcal{A}_p| + 2|\widetilde{\mathcal{N}}_{p-1}| + |\widetilde{\mathcal{A}}_{p-1}| + |\mathcal{N}_p \cap \mathcal{N}^D|$. Especially, we have $\mathcal{K}_p = \varnothing$, if $p = \overline{p}$.

We are now ready to present the formulation for problem $\mathbf{P}_{\xi,p}$, which is an LP model written as follows:

$$\mathbf{P}_{\xi,p}(\bar{\boldsymbol{\chi}}_{\xi,p-1},\Psi_p)=\min_{\boldsymbol{x}_{\xi,p},\eta_{\xi,p}}\sum_{n\in\mathcal{N}_p}(h_n u_{\xi,n}+e_n v_{\xi,n})+\sum_{a\in\mathcal{A}_p}c_a z_{\xi,a}+M\sum_{n\in\mathcal{N}_p\cap\mathcal{N}^D}w_{\xi,n}+\eta_{\xi,p} \tag{21}$$

$$\text{s.t.} \quad (19),(20)$$

$$z_{\xi,a}\le y'_{\xi,p,b} \qquad \forall a\in\mathcal{A}^1_b\cap\mathcal{A}_p,\forall b\in\mathcal{B} \tag{22}$$

$$u_{\xi,n_1}+v_{\xi,n_1}=d_{\xi,n_1}+u_{\xi,n_2}+v_{\xi,n_2}-\sum_{a\in A^+(n_1)}z_{\xi,a} \qquad \forall n_1=(i,t),n_2=(i,t-1)\in\mathcal{N}_p\cap\mathcal{N}^S \tag{23}$$

$$u_{\xi,n_1}+v_{\xi,n_1}=d_{\xi,n_1}+u'_{\xi,p,n_2}+v'_{\xi,p,n_2}-\sum_{a\in A^+(n_1)}z_{\xi,a} \qquad \forall n_1=(i,\underline{t}_p)\in\mathcal{N}_p,\forall n_2=(i,\bar{t}_{p-1})\in\widetilde{\mathcal{N}}_{p-1},\forall i\in\mathcal{I}^S \tag{24}$$

$$u_{\xi,n_1}-v_{\xi,n_1}+w_{\xi,n_1}=d_{\xi,n_1}+u_{\xi,n_2}-v_{\xi,n_2}+\sum_{a\in A^-(n_1)\cap\widetilde{\mathcal{A}}_{p-1}}z'_{\xi,p,a}+\sum_{a\in A^-(n_1)\cap\mathcal{A}_p}z_{\xi,a} \qquad \forall n_1=(i,t),n_2=(i,t-1)\in\mathcal{N}_p\cap\mathcal{N}^D \tag{25}$$

$$u_{\xi,n_1}-v_{\xi,n_1}+w_{\xi,n_1}=d_{\xi,n_1}+u'_{\xi,p,n_2}-v'_{\xi,p,n_2}+\sum_{a\in A^-(n_1)\cap\widetilde{\mathcal{A}}_{p-1}}z'_{\xi,p,a}+\sum_{a\in A^-(n_1)\cap\mathcal{A}_p}z_{\xi,a} \qquad \forall n_1=(i,\underline{t}_p)\in\mathcal{N}_p,\forall n_2=(i,\bar{t}_{p-1})\in\widetilde{\mathcal{N}}_{p-1},\forall i\in\mathcal{I}^D \tag{26}$$

$$u_{\xi,n}\le\bar{q}_n \qquad \forall n\in\mathcal{N}_p \tag{27}$$

$$y'_{\xi,p,b}=\bar{y}_{\xi,p-1,b} \qquad \forall b\in\mathcal{B} \tag{28}$$

$$u'_{\xi,p,n}=\bar{u}_{\xi,p-1,n} \qquad \forall n\in\widetilde{\mathcal{N}}_{p-1} \tag{29}$$

$$v'_{\xi,p,n}=\bar{v}_{\xi,p-1,n} \qquad \forall n\in\widetilde{\mathcal{N}}_{p-1} \tag{30}$$

$$z'_{\xi,p,a}=\bar{z}_{\xi,p-1,a} \qquad \forall a\in\widetilde{\mathcal{A}}_{p-1} \tag{31}$$

$$u_{\xi,n}\ge 0 \qquad \forall n\in\mathcal{N}_p \tag{32}$$

$$v_{\xi,n}\ge 0 \qquad \forall n\in\mathcal{N}_p \tag{33}$$

$$z_{\xi,a}\ge 0 \qquad \forall a\in\mathcal{A}_p \tag{34}$$

$$w_{\xi,n}\ge 0 \qquad \forall n\in\mathcal{N}_p\cap\mathcal{N}^D, \tag{35}$$

where M in the objective function (21) is a sufficiently large constant.

The objective function (21) minimizes the sum of four terms, including (i) the total inventory holding cost and the total backlogging cost at the nodes in $\mathcal{N}_p$, (ii) the total shipping cost for sending flows on the arcs in $\mathcal{A}_p$, (iii) the total penalty cost associated with overflowed inventories at the demand nodes, and (iv) the value of the cost-to-go function. Constraints (22) set upper bounds for the flows on the arcs associated with shipments in the bids. Constraints (23) and (24) track the inventory levels and backlogged supplies at the nodes associated with the supply sites in stage p. Similarly, constraints (25) and (26) track the inventory levels, backlogged demands, and overflowed inventories at the nodes associated with the demand sites in stage p. Constraints (27) require that the inventory stored at each node be maintained under the upper limit. Constraints (28)–(31) link the decision variables determined in the parent problem with their local copies in problem $\mathbf{P}_{\xi,p}$. The last four sets of constraints define the domains of the decision variables.

Finally, let $\boldsymbol{X}^*_{\xi,p}$ be the vector of the solution values of the decision variables in $\boldsymbol{X}_{\xi,p}$ obtained by solving $\mathbf{P}_{\xi,p}$. If $p < \overline{p}$, we obtain $\bar{\chi}_{\xi,p}$, which will be used for defining problem $\mathbf{P}_{\xi,p+1}$, through the following equalities:

$$
\begin{aligned}
\bar{y}_{\xi,p,b} &= y'^{,*}_{\xi,p,b} && \forall b \in \mathcal{B}, \\
\bar{u}_{\xi,p,n} &= u^*_{\xi,n} && \forall n \in \widetilde{\mathcal{N}}_p, \\
\bar{v}_{\xi,p,n} &= v^*_{\xi,n} && \forall n \in \widetilde{\mathcal{N}}_p, \\
\bar{z}_{\xi,p,a} &= z^*_{\xi,a} && \forall a \in \widetilde{\mathcal{A}}_p \cap \mathcal{A}_p, \\
\bar{z}_{\xi,p,a} &= z'^{,*}_{\xi,p,a} && \forall a \in \widetilde{\mathcal{A}}_p \setminus \mathcal{A}_p.
\end{aligned}
$$

5.3. The Backward Step

When all the forward-step problems for each sampled scenario $\xi \in \Xi^l$ are solved in iteration l, the backward step starts from the last stage $p = \overline{p}$. It then moves backward, stage by stage, until reaching stage $p = 1$. In each stage, a set of problems are solved. The goal of the backward step is to update the cost-to-go functions for problems in the forward step.

5.3.1. Problems in Backward Step

In iteration l of the SDDP approach, for each sampled scenario $\xi \in \Xi^l$, we solve $|\Omega_p|$ problems in the backward step in stage $p \in \mathcal{P}$. Each of the problems corresponds to a stage scenario $\omega \in \Omega_p$ in stage p in the *original scenario tree*. Let $\mathbf{Q}_{\xi,\omega,p}$ denote the problem that is associated with scenario $\xi \in \Xi^l$ and stage scenario $\omega \in \Omega_p$ in stage $p \in \mathcal{P}$ in the backward step.

Given $\xi \in \Xi^l$, and $\omega \in \Omega_p$ in stage $p \in \mathcal{P}$, problem $\mathbf{Q}_{\xi,\omega,p}$ and problem $\mathbf{P}_{\xi,p}$ in the forward step are characterized by the same set of state variables obtained by solving the parent problem $\mathbf{P}_{\xi,p-1}$ (or $\mathbf{P}_0$) and the same cost-to-go function (i.e., $\bar{\chi}_{\xi,p-1}$ and Ψ_p).

The decision variables for $\mathbf{Q}_{\xi,\omega,p}$ include those contained in the vector $\boldsymbol{X}_{\omega,p}$ and variable $\eta_{\omega,p}$. Here, there are one-to-one correspondences between variables in $\boldsymbol{X}_{\omega,p}$ and those in $\boldsymbol{X}_{\xi,p}$ of problem $\mathbf{P}_{\xi,p}$ in the forward step. To be more specific, for every variable defined for the scenario ξ in $\boldsymbol{X}_{\xi,p}$, there is a corresponding variable in $\boldsymbol{X}_{\omega,p}$ defined for the stage scenario ω. In addition, $\eta_{\omega,p}$ represents the value returned by the cost-to-go function Ψ_p.

By respectively replacing the variables in $\boldsymbol{X}_{\xi,p}$ and $\eta_{\xi,p}$ and the parameters in $\boldsymbol{d}_{\xi,p}=(d_{\xi,n}|n\in\mathcal{N}_p)$ with their counterparts in $\boldsymbol{X}_{\omega,p}$, $\eta_{\omega,p}$, and $\boldsymbol{d}_{\omega,p}=(d_{\omega,n}|n\in\mathcal{N}_p)$ in constraints (19),(20), (22)–(27) and (32)–(35), problem $\mathbf{Q}_{\xi,\omega,p}$ can be formulated as the following LP model:

$$\mathbf{Q}_{\xi,\omega,p}(\bar{\boldsymbol{\chi}}_{\xi,p-1},\Psi_p)=\min_{\boldsymbol{X}_{\omega,p},\eta_{\omega,p}}\sum_{n\in\mathcal{N}_p}(h_n u_{\omega,n}+e_n v_{\omega,n})+\sum_{a\in\mathcal{A}_p}c_a z_{\omega,a}+M\sum_{n\in\mathcal{N}_p\cap\mathcal{N}^D}w_{\omega,n}+\eta_{\omega,p} \tag{36}$$

$$\text{s.t.}\quad (19),(20),(22)-(27),(32)-(35)$$

$$y'_{\omega,p,b}=\bar{y}_{\xi,p-1,b}\qquad \forall b\in\mathcal{B} \tag{37}$$

$$u'_{\omega,p,n}=\bar{u}_{\xi,p-1,n}\qquad \forall n\in\widetilde{\mathcal{N}}_{p-1} \tag{38}$$

$$v'_{\omega,p,n}=\bar{v}_{\xi,p-1,n}\qquad \forall n\in\widetilde{\mathcal{N}}_{p-1} \tag{39}$$

$$z'_{\omega,p,a}=\bar{z}_{\xi,p-1,a}\qquad \forall a\in\widetilde{\mathcal{A}}_{p-1}. \tag{40}$$

Constraints (37)–(40) link the relevant decision variables in this problem with parameters in $\bar{\boldsymbol{\chi}}_{\xi,p-1}$ which are obtained by solving problem $\mathbf{P}_{\xi,p-1}$ (or $\mathbf{P}_0$).

5.3.2. Update of Cost-to-go Functions In the SDDP, we solve the dual problem of $\mathbf{Q}_{\xi,\omega,p}$, denoted by $\mathbf{D}_{\xi,\omega,p}$, $\forall p\in\mathcal{P},\omega\in\Omega_p,\xi\in\Xi^l$ to generate valid inequalities for updating the cost-to-go functions. In particular, by solving $\mathbf{D}_{\xi,\omega,p}$ to optimality, let $\zeta_{\xi,\omega,p}$ be the optimal objective function value of $\mathbf{D}_{\xi,\omega,p}$, and let $\phi_{\xi,\omega,p,b}$ ($\forall b\in\mathcal{B}$), $\pi_{\xi,\omega,p,n}$ ($\forall n\in\widetilde{\mathcal{N}}_{p-1}$), $\varpi_{\xi,\omega,p,n}$ ($\forall n\in\widetilde{\mathcal{N}}_{p-1}$), and $\theta_{\xi,\omega,p,a}$ ($\forall a\in\widetilde{\mathcal{A}}_{p-1}$) be the optimal solution values of the dual variables associated with constraints (37)–(40), respectively. Given these results, we update the cost-to-go functions as follows.

First, for the cost-to-go function Ψ_0 in stage 0, the following set of inequalities are valid:

$$\eta_0\geq\sum_{\omega\in\Omega_1}\varrho_\omega\zeta_{\xi,\omega,1}+\sum_{\omega\in\Omega_1}\varrho_\omega\sum_{b\in\mathcal{B}}\phi_{\xi,\omega,1,b}(y_b-\bar{y}_{\xi,0,b})\ \forall\xi\in\Xi^l. \tag{41}$$

Let $\mathcal{K}_0^+$ denote the set of these inequalities. We update the cost-to-go function $\Psi_0(\boldsymbol{y})$ by letting $\mathcal{K}_0=\mathcal{K}_0\cup\mathcal{K}_0^+$.

Similarly, for any scenario $\xi'\in\Xi$, we obtain the following set of valid inequalities for the cost-to-go functions Ψ_p in stage $p\in\mathcal{P}\setminus\{\overline{p}\}$:

$$\begin{aligned}\eta_{\xi',p}\geq&\sum_{\omega\in\Omega_{p+1}}\varrho_\omega\zeta_{\xi,\omega,p+1}+\sum_{\omega\in\Omega_{p+1}}\varrho_\omega\sum_{b\in\mathcal{B}}\phi_{\xi,\omega,p+1,b}(y'_{\xi',p,b}-\bar{y}_{\xi,p,b})\\&+\sum_{\omega\in\Omega_{p+1}}\varrho_\omega\sum_{n\in\widetilde{\mathcal{N}}_p}\pi_{\xi,\omega,p+1,n}(u_{\xi',n}-\bar{u}_{\xi,p,n})+\sum_{\omega\in\Omega_{p+1}}\varrho_\omega\sum_{n\in\widetilde{\mathcal{N}}_p}\varpi_{\xi,\omega,p+1,n}(v_{\xi',n}-\bar{v}_{\xi,p,n})\\&+\sum_{\omega\in\Omega_{p+1}}\varrho_\omega\sum_{a\in\widetilde{\mathcal{A}}_p\cap\mathcal{A}_p}\theta_{\xi,\omega,p+1,a}(z_{\xi',a}-\bar{z}_{\xi,p,a})\\&+\sum_{\omega\in\Omega_{p+1}}\varrho_\omega\sum_{a\in\widetilde{\mathcal{A}}_p\setminus\mathcal{A}_p}\theta_{\xi,\omega,p+1,a}(z'_{\xi',p,a}-\bar{z}_{\xi,p,a})\ \forall\xi\in\Xi^l.\end{aligned} \tag{42}$$

Let $\mathcal{K}_p^+$ denote the set of inequalities (42) for problems in stage $p\in\mathcal{P}\setminus\{\overline{p}\}$. We update the cost-to-go functions $\Psi_p(\boldsymbol{X}_{\xi',p})$ by letting $\mathcal{K}_p=\mathcal{K}_p\cup\mathcal{K}_p^+$.

6. Enhancements to the SDDP Approach

In this section, we describe several important enhancements to the SDDP approach proposed in the previous section.

6.1. Feasibility Inequalities

To avoid infeasibilities caused by arc flows between stages, in any iteration l of the SDDP approach, variables $w_{\xi,n}$ and $w_{\omega,n}$ are used in the formulation of problems $\mathbf{P}_{\xi,p}$, and $\mathbf{Q}_{\xi,\omega,p}$, $\omega\in\Omega_p$, $p\in\mathcal{P}$, $\xi\in\Xi^l$ to capture the inventory overages on the demand sites. These variables are penalized with large costs (big-M) in the objective functions. The use of big-M terms leads to poor solutions generated in the forward step and weak cuts for cost-to-go functions obtained in the backward step. To resolve this issue, we propose using valid inequalities to impose the feasibility of the stage-wise problems without using auxiliary variables. Essentially, these inequalities are imposed on problems $\mathbf{P}_{\xi,p}$ and $\mathbf{Q}_{\xi,\omega,p}$ with $\omega\in\Omega_p$, $p\in\mathcal{P}$, and $\xi\in\Xi^l$ to ensure the feasibility of the problems in the related subsequent stages under the most extreme scenario by considering the interconnections between the shipping and inventory decisions among these stages. The feasibility inequalities are derived as follows.

Consider any site $i\in\mathcal{I}^D$. Let $\overline{d}_{i,p_1,t}$ be its minimum amount of demand accumulated from the first period in stage p_1 (i.e., $\underline{t}_{p_1}$) to any period $t\in\mathcal{T}_{p_2}$ where $p_1,p_2\in\mathcal{P}$ and $p_2\geq p_1$ under all scenarios $\xi\in\Xi$. $\overline{d}_{i,p_1,t}$ can be calculated by:

$$\overline{d}_{i,p_1,t}=\sum_{p=p_1}^{p_2-1}\max_{\omega\in\Omega_p}\sum_{t'=\underline{t}_p}^{\overline{t}_p}d^{\omega}_{i,t'}+\max_{\omega\in\Omega_{p_2}}\sum_{t'=\underline{t}_{p_2}}^{t}d^{\omega}_{i,t'}. \tag{43}$$

Using these minimum accumulated demands, we have the following Lemma.

Lemma 1. *The following inequalities are valid for problem* $\mathbf{P}$*:*

$$u_{\xi,n_1}-v_{\xi,n_1}+\sum_{t=\underline{t}_{p+1}}^{t_2}\sum_{n=(i,t)\in\mathcal{N}}\sum_{p'=1}^{p}\sum_{a\in A^-(n)\cap\mathcal{A}_{p'}}z_{\xi,a}+\overline{d}_{i,p+1,t_2}\leq\overline{q}_{n_2}$$
$$\forall n_1=(i,\overline{t}_p),n_2=(i,t_2)\in\mathcal{N}^D,t_2\geq\underline{t}_{p+1},\forall p\in\mathcal{P}\setminus\{\overline{p}\},\forall\xi\in\Xi. \tag{44}$$

Based on Lemma 1, we derive the following inequalities (45) that are valid for problems $\mathbf{P}_{\xi,p}$, where $p\in\mathcal{P}\setminus\{\overline{p}\}$ and $\xi\in\Xi$:

$$u_{\xi,n_1}-v_{\xi,n_1}+\sum_{t=\underline{t}_{p+1}}^{t_2}\sum_{n=(j,t)\in\mathcal{N}}\sum_{a\in A^-(n)\cap\mathcal{A}_p}z_{\xi,a}+\sum_{t=\underline{t}_{p+1}}^{t_2}\sum_{n=(j,t)\in\mathcal{N}}\sum_{a\in A^-(n)\cap\widetilde{\mathcal{A}}_{p-1}}z'_{\xi,p,a}+\overline{d}_{j,p+1,t_2}\leq\overline{q}_{n_2}$$
$$\forall n_1=(j,\overline{t}_p),n_2=(j,t_2)\in\mathcal{N}^D,\forall((i,t_1),(j,t_2))\in\widetilde{\mathcal{A}}_p. \tag{45}$$

Recall that $\widetilde{\mathcal{A}}_{\overline{p}} = \varnothing$. By incorporating inequalities (45) into problems $\mathbf{P}_{\xi,p}$, where $p \in \mathcal{P}$ and $\xi \in \Xi$, we reformulate the problems as

$$\mathbf{P}'_{\xi,p}\ (\bar{\boldsymbol{\chi}}_{\xi,p-1}, \Psi_p) = \min_{\boldsymbol{X}'_{\xi,p}, \eta_{\xi,p}} \sum_{n \in \mathcal{N}_p} (h_n u_{\xi,n} + e_n v_{\xi,n}) + \sum_{a \in \mathcal{A}_p} c_a z_{\xi,a} + \eta_{\xi,p} \tag{46}$$

$$\text{s.t.} \quad (19), (20), (22)-(24), (27)-(34), (45)$$

$$u_{\xi,n_1} - v_{\xi,n_1} = d_{\xi,n_1} + u_{\xi,n_2} - v_{\xi,n_2} + \sum_{a \in A^-(n_1) \cap \widetilde{\mathcal{A}}_{p-1}} z'_{\xi,p,a} + \sum_{a \in A^-(n_1) \cap \mathcal{A}_p} z_{\xi,a}$$
$$\forall n_1 = (i,t), n_2 = (i,t-1) \in \mathcal{N}_p \cap \mathcal{N}^D \tag{47}$$

$$u_{\xi,n_1} - v_{\xi,n_1} = d_{\xi,n_1} + u'_{\xi,p,n_2} - v'_{\xi,p,n_2} + \sum_{a \in A^-(n_1) \cap \widetilde{\mathcal{A}}_{p-1}} z'_{\xi,p,a} + \sum_{a \in A^-(n_1) \cap \mathcal{A}_p} z_{\xi,a}$$
$$\forall n_1 = (i, \underline{t}_p) \in \mathcal{N}_p, \forall n_2 = (i, \bar{t}_{p-1}) \in \widetilde{\mathcal{N}}_{p-1}, \forall i \in \mathcal{I}^D, \tag{48}$$

where $\boldsymbol{X}'_{\xi,p}$ is the vector of all decision variables for $\mathbf{P}'_{\xi,p}$, which is obtained by removing the w variables from $\boldsymbol{X}_{\xi,p}$.

Proposition 2. *Problems $\mathbf{P}'_{\xi,p}$ are always feasible, $\forall p \in \mathcal{P}$, $\forall \xi \in \Xi$.*

Similarly, problems $\mathbf{Q}_{\xi,\omega,p}$ in the backward step can also be reformulated to remove big-M terms in the objective functions. In particular, for any problem $\mathbf{Q}_{\xi,\omega,p}$ in the backward step, where $\omega \in \Omega_p$, $p \in \mathcal{P}$, and $\xi \in \Xi$, let $\boldsymbol{X}'_{\omega,p}$ be the updated vector of decision variables. The vector $\boldsymbol{X}'_{\omega,p}$ contains all decision variables except the w variables in the vector $\boldsymbol{X}_{\omega,p}$. Then, by adapting constraints (19), (20), (22)–(24), (27)–(34), (45), (47), and (48) which are formulated based on variables in $\boldsymbol{X}'_{\xi,p}$ and parameters in $\boldsymbol{d}_{\xi,p}$ to their counterparts formulated based on variables in $\boldsymbol{X}'_{\omega,p}$ and parameters in $\boldsymbol{d}_{\omega,p}$, we reformulate problem $\mathbf{Q}_{\xi,\omega,p}$ as

$$\mathbf{Q}'_{\xi,\omega,p}\ (\bar{\boldsymbol{\chi}}_{\xi,p-1}, \Psi_p) = \min_{\boldsymbol{X}'_{\omega,p}, \eta_{\omega,p}} \sum_{n \in \mathcal{N}_p} (h_n u_{\omega,n} + e_n v_{\omega,n}) + \sum_{a \in \mathcal{A}_p} c_a z_{\omega,a} + \eta_{\omega,p} \tag{49}$$
$$\text{s.t.} \quad (19), (20), (22)-(24), (27)-(34), (37)-(40), (45), (47), (48).$$

Proposition 3. *Problems $\mathbf{Q}'_{\xi,\omega,p}$ are always feasible, $\forall \omega \in \Omega_p$, $\forall p \in \mathcal{P}$, $\forall \xi \in \Xi$.*

With Propositions 2 and 3, in any iteration l of the SDDP approach, we solve problems $\mathbf{P}'_{\xi,p}$ instead of $\mathbf{P}_{\xi,p}$ in the forward step and the dual problems of $\mathbf{Q}'_{\xi,\omega,p}$ instead of $\mathbf{Q}_{\xi,\omega,p}$ in the backward step, where $\omega \in \Omega_p$, $p \in \mathcal{P}$, and $\xi \in \Xi^l$.

6.2. Optimality Inequalities

In the SDDP, the quality of the lower bound depends on the quality of the cost-to-go functions which are obtained by solving problems $\mathbf{Q}_{\xi,\omega,p}$ in the backward step. Note that problems $\mathbf{Q}_{\xi,\omega,p}$ are parameterized by solutions obtained from solving problems $\mathbf{P}_0$ and $\mathbf{P}_{\xi,p}$ in the forward step.

Therefore, having high-quality solutions for problems $\mathbf{P}_0$ and $\mathbf{P}_{\xi,p}$ is critical for generating high-quality cost-to-go functions. However, due to the stage-wise solution procedure, especially in the initial iterations of the SDDP approach, both the lower bound and solutions of problems in the forward step tend to have low quality.

To address these issues, we propose to lift the lower bound and to drive problems $\mathbf{P}_0$ and $\mathbf{P}_{\xi,p}$ to generate high-quality solutions by using optimality inequalities. In particular, for problem $\mathbf{P}_0$ or $\mathbf{P}_{\xi,p}$, the associated optimality inequalities estimate the lower bound of the cost incurred in the subsequent stages, in response to the decisions made in the current stage. Our method of generating optimality inequalities is inspired by Theorem 1 in Chapter 10 of Birge and Louveaux (2011), which derives a valid lower bound of a multi-stage stochastic linear program based on consistent partitions of the stage scenarios. We extend this idea by constructing an *approximate scenario tree* for each stage $p \in \mathcal{P}^+ \setminus \{\overline{p}\}$ based on which the optimality inequalities are formulated.

6.2.1. Approximate Scenario Tree Construction For generating the optimality inequalities, for each stage $p \in \mathcal{P}^+ \setminus \{\overline{p}\}$ in the original scenario tree (*original tree*), we construct an approximate scenario tree (*approximate tree*), denoted by $\mathscr{T}_p$.

Each approximate tree $\mathscr{T}_p$, $p \in \mathcal{P}^+ \setminus \{\overline{p}\}$, consists of a set $\widehat{\mathcal{P}}_p$ of stages, where $\widehat{\mathcal{P}}_p = \{p+1, .., \overline{p}\}$. There is a set $\widehat{\Omega}_k$ of stage scenarios in stage $k \in \widehat{\mathcal{P}}_p$ of the approximate tree $\mathscr{T}_p$. Each stage scenario $\widehat{\omega} \in \widehat{\Omega}_k$ maps a subset of stage scenarios in Ω_k in the original tree, which is denoted by $\Omega_k(\widehat{\omega}) \subseteq \Omega_k$. For any $k \in \widehat{\mathcal{P}}_p$ and $p \in \mathcal{P}^+ \setminus \{\overline{p}\}$, the mapping between $\widehat{\omega} \in \widehat{\Omega}_k$ and $\omega \in \Omega_k$ satisfies:

$$
\begin{aligned}
&\bigcup_{\widehat{\omega} \in \widehat{\Omega}_k} \Omega_k(\widehat{\omega}) = \Omega_k, \\
&\Omega_k(\widehat{\omega}_1) \cap \Omega_k(\widehat{\omega}_2) = \varnothing, \ \ \forall \widehat{\omega}_1, \widehat{\omega}_2 \in \widehat{\Omega}_k, \widehat{\omega}_1 \neq \widehat{\omega}_2.
\end{aligned}
$$

Figure 3 gives an example of generating the approximate tree for stage 1 (i.e., $\mathscr{T}_1$) from the original tree with $|\mathcal{P}| = 3$ and $|\Omega_p| = 6$, $p \in \mathcal{P}$. The tree $\mathscr{T}_1$ is used for formulating the optimality inequalities for any problem $\mathbf{P}_{\xi,1}$.

Given a stage scenario $\widehat{\omega} \in \widehat{\Omega}_k$ in stage $k \in \widehat{\mathcal{P}}_p$ of an approximate tree $\mathscr{T}_p$, we set its realization probability $\widehat{\varrho}_{\widehat{\omega}}$ as $\widehat{\varrho}_{\widehat{\omega}} = \sum_{\omega \in \Omega_k(\widehat{\omega})} \varrho_\omega$. The supply or demand of each node $n \in \mathcal{N}_k$ under stage scenario $\widehat{\omega} \in \widehat{\Omega}_k$, denoted by $\widehat{d}_{\widehat{\omega},n}$, is set as $\widehat{d}_{\widehat{\omega},n} = \sum_{\omega \in \Omega_k(\widehat{\omega})} \frac{\varrho_\omega}{\widehat{\varrho}_{\widehat{\omega}}} d_{\omega,n}$.

For an approximate tree $\mathscr{T}_p$, any path in the form of $\{\widehat{\omega}_{p+1}, ..., \widehat{\omega}_{\overline{p}}\}$ in the tree, where $\widehat{\omega}_k \in \widehat{\Omega}_k$, represents an *approximate scenario*. Let $\widehat{\Xi}_p$ be the set of approximate scenarios associated with $\mathscr{T}_p$. For each $\widehat{\xi} \in \widehat{\Xi}_p$, we denote by $\widehat{\omega}_k(\widehat{\xi}) \in \widehat{\Omega}_k$ the index of the stage scenario in stage $k \in \widehat{\mathcal{P}}_p$ associated with this scenario in the approximate tree $\mathscr{T}_p$. Accordingly, the probability of each scenario $\widehat{\xi} \in \widehat{\Xi}_p$, denoted by $\widehat{\rho}_{\widehat{\xi}}$, is calculated by $\widehat{\rho}_{\widehat{\xi}} = \prod_{k \in \widehat{\mathcal{P}}_p} \widehat{\varrho}_{\widehat{\omega}_k(\widehat{\xi})}$. The supply or demand of any

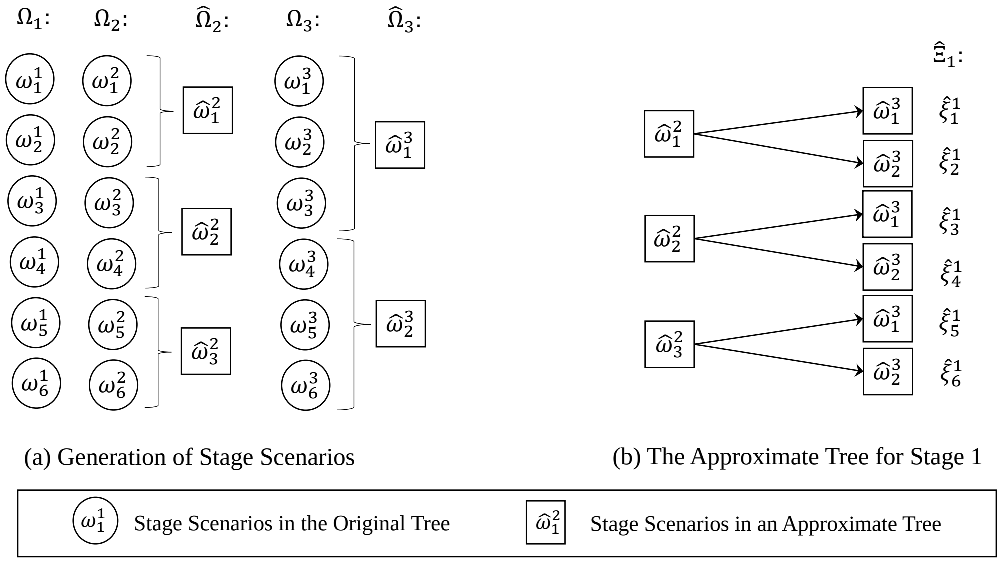

Figure 3 Example of an Approximate Tree.

node $n \in \mathcal{N}_k$ for any $k \in \widehat{\mathcal{P}}_p$ in the scenario tree $\mathscr{T}_p$ under scenario $\widehat{\xi} \in \widehat{\Xi}_p$ is denoted by $\widehat{d}_{\widehat{\xi},n}$ and is set equal to $\widehat{d}_{\widehat{\omega}_k(\widehat{\xi}),n}$. Further, for each scenario tree $\mathscr{T}_p$, $p \in \mathcal{P}^+ \setminus \{\overline{p}\}$, we define the set $\widehat{\Lambda}_{p,k} = \{(\widehat{\xi}_1, \widehat{\xi}_2) \in \widehat{\Xi}_p \times \widehat{\Xi}_p | \widehat{\omega}_{k'}(\widehat{\xi}_1) = \widehat{\omega}_{k'}(\widehat{\xi}_2), \forall k' = p+1, ..., k\}$ which contains all pairs of scenarios $(\widehat{\xi}_1, \widehat{\xi}_2) \in \widehat{\Xi}_p \times \widehat{\Xi}_p$ that are indistinguishable in stage $k \in \widehat{\mathcal{P}}_p$ in the approximate tree.

One can verify that for any stage scenario $\omega \in \Omega_p$, $p \in \mathcal{P}^+ \setminus \{\bar{p}\}$, the approximate tree $\mathscr{T}_p$ forms a consistent partition (Birge and Louveaux 2011) of the stage scenarios in the subtree associated with ω in the original tree. In the next section, we show how to derive optimality inequalities based on these approximate trees.

6.2.2. Deriving Optimality Inequalities The optimality inequalities for problems $\mathbf{P}_0$ and $\mathbf{P}_{\xi,p}$ in the SDDP are generated based on the approximate trees $\mathscr{T}_p$, where $p \in \mathcal{P}^+ \setminus \{\overline{p}\}$ and $\xi \in \Xi$. Given an approximate tree $\mathscr{T}_p$ and its associated approximate scenario set $\widehat{\Xi}_p$ ($p \in \mathcal{P}^+ \setminus \{\overline{p}\}$), the associated optimality inequalities make use of the following additional variables:

- $\widehat{z}_{\widehat{\xi},a}$ continuous variable, which represents the volume of the commodity allocated on arc $a \in \mathcal{A}_k$, $k \in \widehat{\mathcal{P}}_p$ under scenario $\widehat{\xi} \in \widehat{\Xi}_p$;
- $\widehat{u}_{\widehat{\xi},n}$ continuous variable, which represents the inventory level at node $n \in \mathcal{N}_k$, $k \in \widehat{\mathcal{P}}_p$ under scenario $\widehat{\xi} \in \widehat{\Xi}_p$;
- $\widehat{v}_{\widehat{\xi},n}$ continuous variable, which represents the volume of the supply or demand backlogged at node $n \in \mathcal{N}_k$, $k \in \widehat{\mathcal{P}}_p$ under scenario $\widehat{\xi} \in \widehat{\Xi}_p$.

Note that for stage $p=0$, we have $\widehat{\mathcal{P}}_0=\mathcal{P}$. For problem $\mathbf{P}_0$, we have the following valid inequalities:

$$\eta_0 \geq \sum_{\widehat{\xi}\in\widehat{\Xi}_0} \widehat{\rho}_{\widehat{\xi}} \left(\sum_{n\in\mathcal{N}} (h_n \widehat{u}_{\widehat{\xi},n} + e_n \widehat{v}_{\widehat{\xi},n}) + \sum_{a\in\mathcal{A}} c_a \widehat{z}_{\widehat{\xi},a} \right) \tag{50}$$

$$\widehat{z}_{\widehat{\xi},a} \leq y_b \ \forall a\in\mathcal{A}_b^1, \forall b\in\mathcal{B}, \forall\widehat{\xi}\in\widehat{\Xi}_0 \tag{51}$$

$$\widehat{u}_{\widehat{\xi},n_1} + \widehat{v}_{\widehat{\xi},n_1} = \widehat{d}_{\widehat{\xi},n_1} + \widehat{u}_{\widehat{\xi},n_2} + \widehat{v}_{\widehat{\xi},n_2} - \sum_{a\in A^+(n_1)} \widehat{z}_{\widehat{\xi},a} \ \forall n_1=(i,t), n_2=(i,t-1)\in\mathcal{N}^S, \forall\widehat{\xi}\in\widehat{\Xi}_0 \tag{52}$$

$$\widehat{u}_{\widehat{\xi},n} + \widehat{v}_{\widehat{\xi},n} = \widehat{d}_{\widehat{\xi},n} + q_n^0 - \sum_{a\in A^+(n)} \widehat{z}_{\widehat{\xi},a} \ \forall n=(i,1)\in\mathcal{N}^S, \forall\widehat{\xi}\in\widehat{\Xi}_0 \tag{53}$$

$$\widehat{u}_{\widehat{\xi},n_1} - \widehat{v}_{\widehat{\xi},n_1} = \widehat{d}_{\widehat{\xi},n_1} + \widehat{u}_{\widehat{\xi},n_2} - \widehat{v}_{\widehat{\xi},n_2} + \sum_{a\in A^-(n_1)} \widehat{z}_{\widehat{\xi},a} \ \forall n_1=(i,t), n_2=(i,t-1)\in\mathcal{N}^D, \forall\widehat{\xi}\in\widehat{\Xi}_0 \tag{54}$$

$$\widehat{u}_{\widehat{\xi},n} - \widehat{v}_{\widehat{\xi},n} = \widehat{d}_{\widehat{\xi},n} + q_n^0 + \sum_{a\in A^-(n)} \widehat{z}_{\widehat{\xi},a} \ \forall n=(i,1)\in\mathcal{N}^D, \forall\widehat{\xi}\in\widehat{\Xi}_0 \tag{55}$$

$$\widehat{u}_{\widehat{\xi},n} \leq \overline{q}_n \ \forall n\in\mathcal{N}, \forall\widehat{\xi}\in\widehat{\Xi}_0 \tag{56}$$

$$\widehat{z}_{\widehat{\xi}_1,a} = \widehat{z}_{\widehat{\xi}_2,a} \ \forall a\in\mathcal{A}_k, \forall(\widehat{\xi}_1,\widehat{\xi}_2)\in\widehat{\Lambda}_{0,k}, \forall k\in\widehat{\mathcal{P}}_0 \tag{57}$$

$$\widehat{u}_{\widehat{\xi},n_1} - \widehat{v}_{\widehat{\xi},n_1} + \sum_{t=\underline{t}_{p+1}}^{t_2} \sum_{n=(j,t)\in\mathcal{N}} \sum_{a\in A^-(n)} \widehat{z}_{\widehat{\xi},a} + \overline{d}_{j,p+1,t_2} \leq \overline{q}_{n_2}$$
$$\forall n_1=(j,\overline{t}_p), n_2=(j,t_2)\in\mathcal{N}^D, \forall((i,t_1),(j,t_2))\in\widetilde{\mathcal{A}}_p, \forall p\in\mathcal{P}\setminus\{\overline{p}\}, \forall\widehat{\xi}\in\widehat{\Xi}_0 \tag{58}$$

$$\widehat{u}_{\widehat{\xi},n} \geq 0 \ \forall n\in\mathcal{N}, \forall\widehat{\xi}\in\widehat{\Xi}_0 \tag{59}$$

$$\widehat{v}_{\widehat{\xi},n} \geq 0 \ \forall n\in\mathcal{N}, \forall\widehat{\xi}\in\widehat{\Xi}_0 \tag{60}$$

$$\widehat{z}_{\widehat{\xi},a} \geq 0 \ \forall a\in\mathcal{A}, \forall\widehat{\xi}\in\widehat{\Xi}_0. \tag{61}$$

Proposition 4. *The optimality inequalities* (50)–(61) *are valid for problem* $\mathbf{P}_0$.

Moreover, for problems $\mathbf{P}_{\xi,p}$ with $p\in\mathcal{P}\setminus\{\overline{p}\}$ and $\xi\in\Xi$, we have the following valid inequalities:

$$\eta_{\xi,p} \geq \sum_{\widehat{\xi}\in\widehat{\Xi}_p} \widehat{\rho}_{\widehat{\xi}} \left(\sum_{k\in\widehat{\mathcal{P}}_p} \left(\sum_{n\in\mathcal{N}_k} (h_n \widehat{u}_{\widehat{\xi},n} + e_n \widehat{v}_{\widehat{\xi},n}) + \sum_{a\in\mathcal{A}_k} c_a \widehat{z}_{\widehat{\xi},a} \right) \right) \tag{62}$$

$$\widehat{z}_{\widehat{\xi},a} \leq y'_{\xi,p,b} \ \forall a\in\mathcal{A}_b^1\cap\mathcal{A}_k, \forall k\in\widehat{\mathcal{P}}_p, \forall b\in\mathcal{B}, \forall\widehat{\xi}\in\widehat{\Xi}_p \tag{63}$$

$$\widehat{u}_{\widehat{\xi},n_1} + \widehat{v}_{\widehat{\xi},n_1} = \widehat{d}_{\widehat{\xi},n_1} + \widehat{u}_{\widehat{\xi},n_2} + \widehat{v}_{\widehat{\xi},n_2} - \sum_{a\in A^+(n_1)} \widehat{z}_{\widehat{\xi},a}$$
$$\forall n_1=(i,t), n_2=(i,t-1)\in\mathcal{N}_k\cap\mathcal{N}^S, \forall k\in\widehat{\mathcal{P}}_p, \forall\widehat{\xi}\in\widehat{\Xi}_p \tag{64}$$

$$\widehat{u}_{\widehat{\xi},n_1} + \widehat{v}_{\widehat{\xi},n_1} = \widehat{d}_{\widehat{\xi},n_1} + u_{\xi,n_2} + v_{\xi,n_2} - \sum_{a\in A^+(n_1)} \widehat{z}_{\widehat{\xi},a}$$
$$\forall n_2=(i,\overline{t}_p), n_1=(i,\underline{t}_{p+1})\in\mathcal{N}^S, \forall\widehat{\xi}\in\widehat{\Xi}_p \tag{65}$$

$$\widehat{u}_{\widehat{\xi},n_1} - \widehat{v}_{\widehat{\xi},n_1} = \widehat{d}_{\widehat{\xi},n_1} + \widehat{u}_{\widehat{\xi},n_2} - \widehat{v}_{\widehat{\xi},n_2} + \sum_{a\in A^-(n_1)\cap\widetilde{\mathcal{A}}_{p-1}} z'_{\xi,p,a} + \sum_{a\in A^-(n_1)\cap\mathcal{A}_p} z_{\xi,a} + \sum_{k\in\widehat{\mathcal{P}}_p} \sum_{a\in A^-(n_1)\cap\mathcal{A}_k} \widehat{z}_{\widehat{\xi},a}$$

$$\forall n_1 = (i,t), n_2 = (i,t-1) \in \mathcal{N}_k \cap \mathcal{N}^D, \forall k \in \widehat{\mathcal{P}}_p, \forall \widehat{\xi} \in \widehat{\Xi}_p \tag{66}$$

$$\widehat{u}_{\widehat{\xi},n} - \widehat{v}_{\widehat{\xi},n} = \widehat{d}_{\widehat{\xi},n_1} + u_{\xi,n_2} - v_{\xi,n_2} + \sum_{a \in A^-(n_1) \cap \widetilde{\mathcal{A}}_{p-1}} z'_{\xi,p,a} + \sum_{a \in A^-(n_1) \cap \mathcal{A}_p} z_{\xi,a} + \sum_{k \in \widehat{\mathcal{P}}_p} \sum_{a \in A^-(n_1) \cap \mathcal{A}_k} \widehat{z}_{\widehat{\xi},a}$$

$$\forall n_2 = (i, \bar{t}_p), n_1 = (i, \underline{t}_{p+1}) \in \mathcal{N}^D, \forall \widehat{\xi} \in \widehat{\Xi}_p \tag{67}$$

$$\widehat{u}_{\widehat{\xi},n} \le \overline{q}_n \;\; \forall n \in \mathcal{N}_k, \forall k \in \widehat{\mathcal{P}}_p, \forall \widehat{\xi} \in \widehat{\Xi}_p \tag{68}$$

$$\widehat{z}_{\widehat{\xi}_1,a} = \widehat{z}_{\widehat{\xi}_2,a} \;\; \forall a \in \mathcal{A}_k, \forall (\widehat{\xi}_1, \widehat{\xi}_2) \in \widehat{\Lambda}_{p,k}, \forall k \in \widehat{\mathcal{P}}_p \tag{69}$$

$$\widehat{u}_{\widehat{\xi},n_1} - \widehat{v}_{\widehat{\xi},n_1} + \sum_{t=\underline{t}_{k+1}}^{t_2} \sum_{n=(j,t) \in \mathcal{N}} \left(\sum_{a \in A^-(n) \cap \widetilde{\mathcal{A}}_{p-1}} z'_{\xi,p,a} + \sum_{a \in A^-(n) \cap \mathcal{A}_p} z_{\xi,a} + \sum_{k' \in \widehat{\mathcal{P}}_p} \sum_{a \in A^-(n) \cap \mathcal{A}_{k'}} \widehat{z}_{\widehat{\xi},a} \right) + \bar{d}_{j,k+1,t_2} \le \overline{q}_{n_2}$$

$$\forall n_1 = (j, \bar{t}_k), n_2 = (j, t_2) \in \mathcal{N}^D, \forall ((i,t_1),(j,t_2)) \in \widetilde{\mathcal{A}}_k, \forall k \in \widehat{\mathcal{P}}_p \setminus \{\overline{p}\}, \forall \widehat{\xi} \in \widehat{\Xi}_p \tag{70}$$

$$\widehat{u}_{\widehat{\xi},n} \ge 0 \;\; \forall n \in \mathcal{N}_k, \forall k \in \widehat{\mathcal{P}}_p, \forall \widehat{\xi} \in \widehat{\Xi}_p \tag{71}$$

$$\widehat{v}_{\widehat{\xi},n} \ge 0 \;\; \forall n \in \mathcal{N}_k, \forall k \in \widehat{\mathcal{P}}_p, \forall \widehat{\xi} \in \widehat{\Xi}_p \tag{72}$$

$$\widehat{z}_{\widehat{\xi},a} \ge 0 \;\; \forall a \in \mathcal{A}_k, \forall k \in \widehat{\mathcal{P}}_p, \forall \widehat{\xi} \in \widehat{\Xi}_p. \tag{73}$$

PROPOSITION 5. *The optimality inequalities* (62)–(73) *are valid for problem* $\mathbf{P}_{\xi,p}$, *where* $p \in \mathcal{P} \setminus \{\overline{p}\}$ *and* $\xi \in \Xi$.

6.3. Primal-Dual Lifting

In this section, we propose a framework that lifts the cost-to-go functions through inequalities that are generated by iteratively solving the correlated (primal) problems in the forward step and (dual) problems in the backward step. This framework is inspired by the nested Benders decomposition method proposed by Birge (1985).

In any iteration l of the SDDP approach, given a forward-step problem $\mathbf{P}_{\xi,p}$ in stage $p \in \mathcal{P} \setminus \{\overline{p}\}$ under scenario $\xi \in \Xi^l$, let $\eta^*_{\xi,p}$ be the optimal solution value of variable $\eta_{\xi,p}$ of the problem, and let $\zeta^*_{\xi,\omega,p+1}$ be the optimal objective function values of the dual backward-step problems $\mathbf{D}_{\xi,\omega,p+1}$, $\forall \omega \in \Omega_{p+1}$. The primal-dual lifting method strengthens $\mathbf{P}_{\xi,p}$ by iterating between solving problem $\mathbf{P}_{\xi,p}$ and problems $\mathbf{D}_{\xi,\omega,p+1}$ ($\omega \in \Omega_{p+1}$) to generate inequalities (42) for the cost-to-go function Ψ_p until a *local convergence* is reached such that we have:

$$\eta^*_{\xi,p} \ge (1-\epsilon) \sum_{\omega \in \Omega_{p+1}} \varrho_\omega \zeta^*_{\xi,\omega,p+1}, \tag{74}$$

where $\epsilon \in [0,1]$ is a preset parameter.

Note that convergence is guaranteed due to the limited number of extreme points of the polyhedral feasible regions for the problems $\mathbf{D}_{\xi,\omega,p+1}$ (Benders 1962).

Similarly, this lifting procedure is also used to lift Ψ_0 of problem $\mathbf{P}_0$ in the SDDP. However, solving MILP models can be time-consuming. To speed this up, in the primal-dual lifting procedure for $\mathbf{P}_0$, we fix the value of decision variables x_b at x_b^*, for each $b \in \mathcal{B}$. Here, x_b^* represents the optimal values of variables x_b returned by solving $\mathbf{P}_0$ before the lifting procedure. Hence, $\mathbf{P}_0$ is solved as an LP model in the lifting procedure.

7. Computational Experiments

We have performed extensive computational experiments to confirm the applicability and effectiveness of our model and algorithm. In this section, we first introduce the experimental settings in Section 7.1. Section 7.2 explains how the testing instances were generated. We then present the computational results, which consist of two parts. In the first part (Section 7.3), we examine the impacts of the enhancement techniques on the performance of the SDDP approach. In the second part (Section 7.4), we compare the performance of the approach with that of other commonly used solution methods. Interested readers can find our code implementation, data sets used, detailed results, and associated user instructions at a link to be provided after the paper is accepted.

7.1. Computational Settings

In order to provide a thorough computational assessment of our proposed SDDP approach, we have implemented the following four variants of the SDDP approach:

1. S0 solves the problem using the basic SDDP approach proposed in Section 5;
2. S1 is similar to S0 but also uses the feasibility inequalities of Section 6.1;
3. S2 is similar to S1 but also uses the optimality inequalities of Section 6.2;
4. S3 is similar to S2 but also uses the primal-dual lifting strategy of Section 6.3.

7.1.1. Implementation Details To alleviate the computational burden of solving the MILP model of problem $\mathbf{P}_0$ in the SDDP approach, all variants of the approach were implemented in a three-phase framework. In the first phase, the integrality constraints of $\mathbf{P}_0$ are dropped. In the second phase, integrality constraints on $\mathbf{P}_0$ are imposed. In the last phase, we solve problem $\mathbf{P}_0$ with the (final) updated cost-to-go function Ψ_0 to obtain the final lower bound and solutions to $\mathbf{P}_0$.

We set the sample size $|\Xi^l| = 16$ for any iteration l in the SDDP approaches and the value of parameter ϵ used in the primal-dual lifting in S3 was set to 1%. To construct the approximate trees in S2 and S3, we let

$$|\widehat{\Omega}_k| = \begin{cases} 2, \text{ if } \; k \in \widehat{\mathcal{P}}_0 \text{ and } k \leq 3, \\ 2, \text{ if } \; k = p+1 \in \widehat{\mathcal{P}}_p, \forall p \in \mathcal{P} \setminus \{\overline{p}\}, \\ 1, \text{ otherwise.} \end{cases}$$

The stopping criteria are as follows. The first (resp. second) phase terminates when (i) the improvement of the lower bound (LB) between adjacent iterations is within 0.1% in any 10 successive iterations, which indicates the convergence of a phase in the approach or (ii) the computational time reaches 30 (resp. 100) minutes. The time limit for solving the MILP model of $\mathbf{P}_0$ in any iteration of any phase is set to 20 minutes.

We implemented our algorithms in C++, and all the experiments were conducted on the Cedar cluster of Compute Canada with 16GB of RAM in a Linux environment. We used CPLEX 12.6.3 for solving the MILP and LP models.

In each iteration l of the SDDP approach, the method solves $|\Xi^l|$ independent problems in the forward step in each stage $p \in \mathcal{P}$. Meanwhile, for each sampled scenario $\xi \in \Xi^l$, $|\Omega_p|$ independent problems are solved in each stage $p \in \mathcal{P}$ in the backward step. To speed up our approach, we solve multiple independent problems simultaneously by using parallel computing. In the experiments, the independent problems were solved in a 16-thread environment (i.e., at most 16 independent problems were solved simultaneously). Note that the solutions of the independent problems solved in each stage in the backward step are synchronized to update the cost-to-go function for the corresponding forward-step problems. Meanwhile, we also let CPLEX run on 16 threads when solving the MILP model of problem $\mathbf{P}_0$ in the SDDP.

7.1.2. Lower Bounds and Upper Bounds In order to evaluate the performance of an approach, we derive the lower bound (LB) and upper bound (UB) obtained by the approach for an instance as follows.

Let Z_0^* and δ be the (sub)optimal objective function value and the optimality gap obtained by solving problem $\mathbf{P}_0$ in the third phase of the SDDP approach. Let also $\boldsymbol{y}^*$ be the solutions of y_b variables obtained by solving this problem.

The lower bound of the instance is calculated as $LB = Z_0^*(1-\delta)$. To obtain the upper bound, a sample set $\Xi' \subseteq \Xi$ is created. If $|\Xi| \leq 10,000$, we let $\Xi' = \Xi$. Otherwise, we independently and randomly sample 10,000 scenarios from Ξ to construct Ξ'. The probability of each scenario $\xi \in \Xi'$ is set as $\rho'_\xi = \frac{\rho_\xi}{\sum_{\xi \in \Xi'} \rho_\xi}$. Then, for each scenario $\xi \in \Xi'$ we solve problems $\mathbf{P}'_{\xi,p}$ in each stage $p \in \mathcal{P}$ with the given $\boldsymbol{y}^*$. For deriving the upper bounds obtained by approaches S0 and S1, these problems are solved without the optimality inequalities (62)–(73) while we solve problems with those inequalities for deriving the upper bounds obtained by approaches S2 and S3.

Let $\gamma^*_{\xi,p}$ and $\eta^*_{\xi,p}$ be the optimal objective function value and the optimal solution value of $\eta_{\xi,p}$ obtained by solving problem $\mathbf{P}'_{\xi,p}$, respectively. We let $Z^*_{\xi,p} = \gamma^*_{\xi,p} - \eta^*_{\xi,p}$, which represents the total cost associated with the decisions made in stage p under scenario ξ. Let also $\mu_\xi = \sum_{p \in \mathcal{P}} Z^*_{\xi,p}$, $\hat{\mu} = \sum_{\xi \in \Xi'} \rho'_\xi \mu_\xi$, and $\sigma^2 = \frac{1}{|\Xi'|-1} \sum_{\xi \in \Xi'} (\mu_\xi - \hat{\mu})^2$.

Finally, for the case with $|\Xi| \leq 10,000$, we set $UB = \sum_{b \in \mathcal{B}} F_b y_b^* + \hat{\mu}$, which is the "true" upper bound for the instance. For the case with $|\Xi| > 10,000$, we set $UB = \sum_{b \in \mathcal{B}} F_b y_b^* + \hat{\mu} + 1.96 \frac{\sigma^2}{\sqrt{|\Xi'|}}$, which represents a 95%-confidence statistical upper bound for the instance. Given LB and UB, the optimality gap of the instance is calculated by $GAP = 100(UB - LB)/LB$.

7.2. Instance Generation

To test the performance of the SDDP approaches, we used 150 instances. These instances were generated based on 10 *cases*. Each case represents a deterministic FPTMP instance (i.e., an SFPTMP instance with a sole scenario). Of the 10 cases, five were adapted from the instances originally created by Papageorgiou et al. (2014) for the maritime inventory routing problem (MIRP) and the other five were generated based on the real manufacturing and iron ore transportation data from a large steel manufacturer. We refer to the five MIRP-based cases as type-I cases and those based on the real data as type-II cases.

In each case, a period represents one week. For generating the SFPTMP instances, we let each stage contain six periods (i.e., six weeks). Given each case, we generated 15 SFPTMP instances. In each instance, we let the number of stages $|\mathcal{P}| \in \{3, 6, 9\}$ which represents planning horizons of 18, 36, and 54 weeks in practice. For each instance, we set the number of stage scenarios at each stage $|\Omega_p| = 10$. The stage scenarios in any Ω_p $(p \in \mathcal{P})$ were generated by a Monte-Carlo simulation in which the demand of each demand site $i \in \mathcal{I}^D$ in each period $t \in \mathcal{T}$ is independent and generated through the uniform distribution $U[\bar{d}_{it}(1-\Delta), \bar{d}_{it}(1+\Delta)]$, where $\bar{d}_{it}$ is the *nominal demand* in the case and $\Delta \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$ is a selected deviation ratio.

Details of the cases and the settings of other parameters in the SFPTMP instances are explained in Appendix C.

7.3. Impacts of Enhancements

To investigate the impacts of the enhancement strategies proposed in Section 6, we have implemented approaches S0, S1, S2, and S3 to solve the instances. The results are reported in Table 2. In this table, the results of five instances that were generated based on the same type of cases and share the same Δ and $|\mathcal{P}|$ are reported as a group. The first two columns show the type of cases and the settings of Δ and $|\mathcal{P}|$ of an instance group. Columns *LB*, *UB*, *GAP* report the average lower bound, the average upper bound, and the average optimality gap generated by each solution method for solving the instances in a group, respectively. In these columns, we use boldface to indicate the best results. Finally, the average computational times are presented in column *TIME*.

The feasibility inequalities remove big-M parameters from the formulation. By comparing the results of the approaches S0 and S1 in Table 2, we can see that for most instances, the usage of these inequalities leads to a dramatic decrease in the optimality gaps. Besides, the optimality inequalities improve the lower bounds and upper bounds and also contribute to substantial decreases in the optimality gaps. Finally, we can see that S3, which uses the primal-dual lifting technique, reports the best lower bounds and secures the smallest optimality gaps for most instance groups. The

Table 2 Computational Results of the SDDP Algorithms.

ST	$\|\mathcal{P}\|$	Δ	LB ($\times 10^3$)				UB ($\times 10^3$)				GAP (%)				TIME (s)			
			S0	S1	S2	S3	S0	S1	S2	S3	S0	S1	S2	S3	S0	S1	S2	S3
I	3	0.1	174.1	199.0	200.5	**200.6**	233.6	204.0	**201.7**	**201.7**	38.9	2.5	**0.6**	**0.6**	54.5	40.7	57.1	105.4
I	3	0.2	181.8	199.9	201.0	**201.3**	223.1	204.5	**203.0**	**203.0**	26.7	2.3	1.0	**0.9**	63.9	33.7	47.1	151.6
I	3	0.3	192.7	201.5	202.2	**202.7**	216.8	206.1	**205.1**	205.4	16.6	2.4	1.4	**1.3**	94.3	40.9	49.1	192.5
I	3	0.4	194.3	202.7	202.9	**203.7**	214.3	206.3	206.8	**205.7**	13.0	1.8	2.0	**1.0**	94.2	35.1	47.4	186.7
I	3	0.5	211.8	212.8	212.9	**213.8**	218.0	217.3	**216.9**	**216.9**	3.0	2.1	1.9	**1.5**	114.6	40.9	48.8	264.4
Average			190.9	203.2	203.9	**204.4**	221.2	207.6	206.7	**206.5**	19.6	2.2	1.4	**1.1**	84.3	38.3	49.9	180.1
I	6	0.1	273.8	325.0	328.5	**328.6**	425.2	339.0	**330.9**	331.0	67.9	4.2	**0.8**	**0.8**	411.0	186.1	86.0	151.4
I	6	0.2	249.1	326.5	329.3	**330.1**	497.0	338.0	334.5	**334.3**	107.2	3.6	1.6	**1.3**	130.0	269.7	167.7	1079.0
I	6	0.3	293.7	333.6	335.8	**337.3**	425.5	345.6	**342.5**	342.7	57.3	3.6	2.0	**1.6**	596.4	207.0	168.3	612.9
I	6	0.4	298.4	337.2	337.7	**340.0**	421.6	347.9	347.0	**346.4**	49.0	3.2	2.8	**1.9**	531.2	240.3	144.9	772.2
I	6	0.5	310.3	336.6	336.6	**339.5**	376.0	348.1	347.4	**345.5**	30.2	3.4	3.2	**1.8**	562.9	175.4	177.4	818.5
Average			285.1	331.8	333.6	**335.1**	429.1	343.7	340.5	**340.0**	62.3	3.6	2.1	**1.5**	446.3	215.7	148.8	686.8
I	9	0.1	263.6	459.3	466.1	**466.4**	698.6	482.1	**469.7**	470.0	214.9	5.0	**0.8**	**0.8**	453.3	572.6	194.0	438.0
I	9	0.2	241.8	462.9	468.3	**469.1**	854.7	482.7	**475.5**	475.7	267.3	4.3	1.6	**1.4**	218.9	427.7	350.3	689.0
I	9	0.3	317.3	468.0	472.5	**475.1**	691.1	490.6	485.0	**484.7**	140.3	4.9	2.7	**2.0**	537.3	530.0	496.7	1570.4
I	9	0.4	349.1	475.6	477.0	**481.0**	676.7	494.7	493.4	**492.3**	111.8	4.1	3.5	**2.4**	464.0	473.4	419.8	1377.2
I	9	0.5	325.8	482.2	482.6	**487.6**	711.4	500.5	501.8	**497.6**	139.9	3.8	4.0	**2.1**	609.6	425.0	296.3	1239.9
Average			299.5	469.6	473.3	**475.8**	726.5	490.1	485.1	**484.0**	174.9	4.4	2.5	**1.8**	456.6	485.7	351.4	1062.9
II	3	0.1	**24.0**	**24.0**	**24.0**	**24.0**	25.0	24.2	**24.0**	**24.0**	4.6	0.8	**0.2**	**0.2**	86.3	73.0	25.3	52.7
II	3	0.2	**24.2**	**24.2**	**24.2**	**24.2**	**24.2**	**24.2**	**24.2**	**24.2**	**0.0**	**0.0**	0.3	0.3	63.5	72.2	28.1	52.1
II	3	0.3	25.2	**25.3**	**25.3**	**25.3**	**25.3**	25.7	**25.3**	**25.3**	0.4	1.5	0.3	**0.1**	76.6	86.0	39.1	62.2
II	3	0.4	**25.2**	**25.2**	**25.2**	**25.2**	**25.2**	**25.2**	**25.2**	**25.2**	**0.0**	**0.0**	0.1	0.1	79.5	77.9	48.5	70.3
II	3	0.5	**24.6**	**24.6**	**24.6**	**24.6**	**24.6**	**24.6**	24.8	**24.6**	**0.0**	**0.0**	0.6	**0.0**	61.5	58.3	53.6	70.6
Average			**24.6**	**24.6**	**24.6**	**24.6**	24.9	24.8	**24.7**	**24.7**	1.0	0.5	0.3	**0.2**	73.5	73.5	38.9	61.6
II	6	0.1	74.1	**74.4**	**74.4**	**74.4**	76.3	75.9	**74.6**	**74.6**	3.0	2.0	**0.2**	**0.2**	231.5	304.1	54.2	67.9
II	6	0.2	75.4	75.4	**75.9**	**75.9**	79.3	80.9	**76.4**	**76.4**	5.0	7.3	**0.7**	**0.7**	297.9	373.9	87.9	114.5
II	6	0.3	76.1	76.1	76.0	**76.2**	77.0	78.4	76.4	**76.3**	1.1	3.1	0.5	**0.2**	259.7	291.9	105.4	147.1
II	6	0.4	79.7	79.8	80.0	**80.1**	83.5	84.7	**80.3**	**80.3**	4.7	6.1	**0.3**	**0.3**	270.6	295.1	123.7	220.2
II	6	0.5	**77.9**	77.8	77.8	77.8	**78.0**	78.2	78.4	78.3	**0.1**	0.5	0.8	0.6	244.0	205.5	172.9	189.0
Average			76.7	76.7	76.8	**76.9**	78.8	79.6	**77.2**	**77.2**	2.8	3.8	0.5	**0.4**	260.7	294.1	108.8	147.8
II	9	0.1	118.7	**119.3**	119.2	119.2	124.4	122.5	**119.6**	**119.6**	5.0	2.8	**0.4**	**0.4**	549.5	689.7	128.5	157.3
II	9	0.2	119.1	119.3	120.6	**120.7**	127.3	131.9	**121.4**	**121.4**	6.8	10.5	**0.6**	**0.6**	603.9	652.1	140.0	243.9
II	9	0.3	120.6	119.9	**122.2**	122.1	139.1	133.9	123.9	**123.7**	15.2	11.8	1.4	**1.2**	654.5	556.4	306.6	319.1
II	9	0.4	125.7	125.3	127.5	**127.6**	135.9	143.3	129.2	**128.8**	8.3	14.7	1.4	**0.9**	782.2	675.5	477.9	642.3
II	9	0.5	120.8	120.5	**121.7**	**121.7**	135.7	125.2	124.0	**122.7**	12.0	4.0	1.9	**0.8**	757.0	697.9	499.1	746.8
Average			121.0	120.9	122.2	**122.3**	132.5	131.4	123.6	**123.2**	9.5	8.8	1.1	**0.8**	669.4	654.3	310.4	421.9

technique is of greater value for instances with longer planning horizons and greater uncertainties in demand and supply.

As for the solution time, the SDDP approach converges more quickly after the incorporation of feasibility inequalities and optimality inequalities. However, the use of the primal-dual lifting technique does slow down the solution procedure. Nevertheless, as a tactical problem, the SFPTMP is solved only once during a long period (several months or a year), such computational times (all less than 30 minutes) are acceptable.

7.4. Comparisons with Alternative Methods

We next compare the performance of the SDDP approach with that of three alternative solution methods, including a commonly used optimization solver (CPLEX) solving the MILP model and two benchmark methods that simulate common decision policies used in practice.

7.4.1. Comparisons with CPLEX We adopted CPLEX (under the computational settings described in Section 7.1.1) on the MILP model of problem **P** to solve the instances. Table 3 summarizes the results produced by CPLEX and S3. We first measure the sizes of the MILP model for the instances in different groups by reporting the average numbers of variables and constraints in columns *#V* and *#C*, respectively. Column *UBG* reports the average gaps (in percentage) of the upper bounds obtained by S3 against those obtained by CPLEX. Because there are more than 4.3×10^8 variables and 6.8×10^9 constraints in any instance with $|\mathcal{P}| \geq 6$, CPLEX can only solve instances with three stages and it is unable to produce the results for larger instances due to memory issue.

Table 3 Computational Results of CPLEX and S3.

ST	$\|\mathcal{P}\|$	Δ	Model Size		LB ($\times 10^3$)		UB ($\times 10^3$)		GAP (%)		TIME (s)		UBG (%)
			#V ($\times 10^3$)	#C ($\times 10^3$)	CPLEX	S3	CPLEX	S3	CPLEX	S3	CPLEX	S3	
I	3	0.1	793.5	10085.8	201.1	200.6	201.1	201.7	0.0	0.6	1815.7	105.4	0.3
I	3	0.2	795.7	10253.6	201.9	201.3	201.9	203.0	0.0	0.9	1416.2	151.6	0.6
I	3	0.3	796.1	10146.9	203.7	202.7	203.7	205.4	0.0	1.3	1932.8	192.5	0.8
I	3	0.4	794.3	10118.1	204.7	203.7	204.7	205.7	0.0	1.0	1572.0	186.7	0.5
I	3	0.5	795.7	10241.9	215.0	213.8	215.0	216.9	0.0	1.5	1508.8	264.4	0.9
Average			795.1	10169.3	205.3	204.4	205.3	206.5	0.0	1.1	1649.1	180.1	0.6
II	3	0.1	433.5	6971.8	24.0	24.0	24.0	24.0	0.0	0.2	600.3	52.7	0.1
II	3	0.2	432.9	6968.5	24.2	24.2	24.2	24.2	0.0	0.3	455.2	52.1	0.3
II	3	0.3	430.7	6812.4	25.3	25.3	25.3	25.3	0.0	0.1	339.0	62.2	0.1
II	3	0.4	432.1	6946.1	25.2	25.2	25.2	25.2	0.0	0.1	370.6	70.3	0.1
II	3	0.5	432.3	6920.2	24.6	24.6	24.6	24.6	0.0	0.0	328.3	70.6	0.0
Average			432.3	6923.8	24.6	24.6	24.6	24.7	0.0	0.2	418.7	61.6	0.1

The results in Table 3 indicate that CPLEX can solve small-scale instances, obtaining optimal solutions in reasonable computational times. Meanwhile, the SDDP approach can also obtain high-quality solutions to these instances but in significantly shorter computational times. Furthermore, as shown in Table 2, for instances that are out of the capacity of CPLEX, the SDDP approach serves as a highly reliable and efficient alternative.

7.4.2. Comparisons with Other Benchmark Methods We have also compared the performance of the SDDP approach with that of two benchmark solution methods. The first method (BM1) simulates a decision policy ignoring capacity contracts, and the second method (BM2) is a deterministic solution construction approach that simulates a myopic decision policy. In particular, when applying BM1 to solve an instance, we run the SDDP approach (i.e., S3) in which the x

variables in problem $\mathbf{P}_0$ are set to zero. Besides, when applying BM2 to solve an instance, we first solve a deterministic version of problem $\mathbf{P}$ which has a sole scenario defined by the average demands and supplies. Then, in each stage $p \in \mathcal{P}$, we solve a problem that is defined solely by the state variables obtained in previous stages and the observed stage scenario (without any cost-to-go functions). Note that to ensure feasibility, the feasibility inequalities (defined in Section 6.1) are incorporated in problems in stages $p \in \mathcal{P}$.

For an instance, the upper bound obtained by BM1 or BM2 is derived using a large set of scenarios generated by the method as described in Section 7.1.2. To evaluate the savings generated by the SDDP approach, in Table 4 we report the average gaps of the upper bounds produced by S3 against those produced by BM1 and BM2, respectively, in columns *UBG1* and *UBG2*.

Table 4 Improvements Generated by SDDP Against the Benchmark Methods.

ST	$\|\mathcal{P}\|$	Δ	UBG1(%)	UBG2(%)	ST	$\|\mathcal{P}\|$	Δ	UBG1(%)	UBG2(%)
I	3	0.1	19.8	31.9	II	3	0.1	31.2	3.9
I	3	0.2	19.1	26.7	II	3	0.2	31.1	11.0
I	3	0.3	18.9	27.3	II	3	0.3	29.1	15.5
I	3	0.4	18.2	25.0	II	3	0.4	28.9	11.5
I	3	0.5	16.7	25.5	II	3	0.5	30.4	11.6
Average			18.5	27.3	Average			30.1	10.7
I	6	0.1	24.3	35.3	II	6	0.1	27.8	9.6
I	6	0.2	23.6	37.4	II	6	0.2	25.7	33.6
I	6	0.3	22.4	42.6	II	6	0.3	25.7	15.6
I	6	0.4	21.6	37.1	II	6	0.4	23.3	30.4
I	6	0.5	21.5	39.4	II	6	0.5	24.3	25.1
Average			22.7	38.3	Average			25.4	22.9
I	9	0.1	26.2	45.2	II	9	0.1	26.5	27.9
I	9	0.2	25.6	38.3	II	9	0.2	25.5	21.5
I	9	0.3	24.4	36.1	II	9	0.3	24.3	31.7
I	9	0.4	23.8	41.7	II	9	0.4	21.3	36.9
I	9	0.5	23.2	39.8	II	9	0.5	24.9	23.6
Average			24.7	40.2	Average			24.5	28.3

As we can see from Table 4, compared to transporting all commodities using the standard freight rates (as in BM1), securing long-term capacities with the carriers helps reduce the total cost for a shipper by 19% to 31%. Besides, unlike the SDDP approach, in BM2, the SFPTMP is solved by overlooking the distributions of uncertain demands and supplies and the interconnection between decision stages. Therefore, the outperformance of S3 against BM2 demonstrates the value of multi-stage stochastic optimization for solving the SFPTMP.

The cost distributions in the solutions generated by the methods are presented in Table 5. In this table, we report the distribution of the cost components in the average total cost obtained by each method for solving instances with the same type of generation cases and $|\mathcal{P}|$. Columns *Bidding*, *Standard Shipping*, *Inventory*, and *Backlog* report the percentages of the costs associated with procuring capacities from the bids, transporting through the standard rates, holding inventories, and managing backlogs, respectively, in the total costs. Note that for instances generated from the first type of cases, we have zero unit inventory costs at all sites, which explains the zero values in column *Inventory* for these instances.

Table 5 Cost Distributions Generated by the Benchmark Methods and S3.

ST	$\|\mathcal{P}\|$	Bidding(%)			Standard Shipping(%)			Inventory(%)			Backlog(%)		
		S3	BM1	BM2	S3	BM1	BM2	S3	BM1	BM2	S3	BM1	BM2
I	3	44.3	0.0	31.0	19.9	71.9	20.0	0.0	0.0	0.0	35.8	28.1	49.0
I	6	54.1	0.0	32.5	20.1	81.1	19.4	0.0	0.0	0.0	25.8	18.9	48.1
I	9	59.6	0.0	35.6	18.3	84.3	16.7	0.0	0.0	0.0	22.1	15.7	47.7
Average		52.7	0.0	33.1	19.4	79.1	18.7	0.0	0.0	0.0	27.9	20.9	48.3
II	3	78.0	0.0	55.6	18.2	97.8	9.9	0.7	0.3	0.5	3.1	1.8	13.0
II	6	63.4	0.0	47.3	23.6	91.0	17.8	0.4	0.1	0.4	12.6	8.8	45.4
II	9	63.7	0.0	49.1	23.2	91.0	16.8	0.3	0.1	0.5	12.7	8.8	71.4
Average		68.4	0.0	50.7	21.7	93.3	14.8	0.4	0.2	0.5	9.5	6.5	43.3

As is shown in Table 5, S3 generates the highest proportions in the total costs for securing capacities from the bids, indicating that it achieves the highest efficiency in freight procurement. Meanwhile, because of the high standard freight rates, shipping contributes 71.9% to 97.8% of the total costs obtained by BM1. Finally, we can see that BM2, which overlooks interactions between the decision stages, is associated with the highest proportions of inventory and backlogging costs.

We have further analyzed the structures of the solutions generated by the benchmark methods and S3. In particular, given any solution of an instance, we calculate the average per-period aggregate distribution volume in all shipments, that in shipments under capacity contracts, and that in shipments acquired through the standard freight rates. We also calculate the average per-period aggregate inventory level at all supply sites and that at all demand sites as well as the average per-period aggregate backlog at all supply sites and that at all demand sites. In Tables 6 to 8 we report the averages of these values derived from the solutions generated by the methods for each set of instances with the same type of generation cases and $|\mathcal{P}|$.

Table 6 Shipment Volumes Generated by the Benchmark Methods and S3.

ST	$\|\mathcal{P}\|$	Total			Capacity Contracts			Standard Shipping		
		S3	BM1	BM2	S3	BM1	BM2	S3	BM1	BM2
I	3	1864.1	1828.5	1837.2	1455.4	0.0	1321.8	408.7	1828.5	515.4
I	6	1815.6	1795.3	1800.4	1441.0	0.0	1303.7	374.6	1795.3	496.8
I	9	1830.7	1816.3	1821.7	1483.8	0.0	1368.5	346.9	1816.3	453.1
Average		1836.8	1813.4	1819.8	1460.1	0.0	1331.4	376.7	1813.4	488.4
II	3	65.7	64.8	65.7	57.3	0.0	57.9	8.4	64.8	7.8
II	6	88.4	87.6	87.0	70.3	0.0	66.5	18.1	87.6	20.4
II	9	93.0	91.9	91.0	74.8	0.0	70.7	18.2	91.9	20.3
Average		82.4	81.4	81.2	67.5	0.0	65.1	14.9	81.4	16.2

We can see from Table 6 that the total shipment volumes generated by all methods are similar. S3 reports the highest proportions of shipment volumes under capacity contracts, followed by BM2

Table 7 Inventory Levels Generated by the Benchmark Methods and S3.

ST	$\|\mathcal{P}\|$	Supply Sites			Demand Sites		
		S3	BM1	BM2	S3	BM1	BM2
I	3	1762.5	1533.4	2076.2	921.4	1820.5	699.5
I	6	1640.0	1354.7	2080.3	1032.6	1954.6	766.4
I	9	1554.6	1286.4	2030.2	1121.6	2016.2	840.0
Average		1652.4	1391.5	2062.2	1025.2	1930.4	768.6
II	3	90.0	66.8	106.4	101.7	130.9	94.5
II	6	80.8	33.0	121.1	98.0	147.2	84.7
II	9	66.6	24.2	136.5	106.0	150.1	79.5
Average		79.1	41.3	121.3	101.9	142.7	86.2

Table 8 Backlog Levels Generated the Benchmark Methods and S3.

ST	$\|\mathcal{P}\|$	Supply Sites			Demand Sites		
		S3	BM1	BM2	S3	BM1	BM2
I	3	1481.6	887.1	1828.9	559.6	535.2	878.3
I	6	1264.4	650.5	1652.1	412.3	378.4	894.4
I	9	1204.6	598.6	1575.2	369.9	348.2	886.4
Average		1316.9	712.1	1685.4	447.3	420.6	886.4
II	3	0.0	0.0	0.0	0.7	0.6	4.2
II	6	0.0	0.0	0.0	4.5	4.2	21.7
II	9	0.0	0.0	0.0	4.8	4.5	36.0
Average		0.0	0.0	0.0	3.3	3.1	20.6

and then BM1. As shown in Tables 7 and 8, BM1 generates the lowest backlog levels for all groups of instances and it also generates the lowest inventory levels at the supply sites for instances with type-II generation cases (where inventories at the supply sites incur non-zero costs). This is because compared to capacity contracts, shipments acquired through the standard freight rates are more flexible. Hence, commodities can be transported from the supply side to the demand side in a more timely manner, avoiding the costly inventories or backlogs on both sides. Finally, we can also find that the myopic solution policy (BM2) leads to the highest backlog levels on both sides.

8. Conclusions

In this study, we have introduced an SFPTMP in the supply chain management of a shipper that sources freight services from the 3PL carriers. We have formulated the problem as a multi-stage stochastic programming model and have developed an SDDP approach for solving the model. To improve the performance of the approach, we have derived feasibility inequalities and optimality inequalities for the stage-wise problems and have proposed a primal-dual lifting procedure.

Extensive computational experiments have been performed. The results show that the enhancement strategies can significantly improve the performance of the approach and that the approach can obtain near-optimal solutions to instances of realistic scale. We have also compared the performance of the SDDP approach and other solution methods and the results attest to the greater effectiveness and efficiency of SDDP against its counterparts.

The current study can be extended in several ways. While this study assumes that the probabilistic distribution of the uncertain parameters is available, this distribution can be unknown in practice especially due to limited data. Hence, it would be interesting to develop robust optimization methods for solving the FPTMP under uncertainty such that the distribution information of uncertain parameters is not fully available. Another possible extension is a consideration of short-term spot market rates in addition to standard shipping rates. Such spot market rates are stochastic and actual spot rates become available shortly prior to shipments. The SDDP approach can be adapted to solve instances with uncertain standard freight rates. Therefore, it would be interesting to further investigate the impacts of uncertain standard freight rates on the performance of the SDDP approach and the structures of the solutions to the SFPTMP.

References

Akarca O (2018) An Overview of 3PL Logistics. https://www.morethanshipping.com/an-overview-of-3pl-logistics/, Accessed March 1, 2022.

Allied Market Research (2019) Third-party logistics (3PL) market. https://www.alliedmarketresearch.com/3PL-market, Accessed March 1, 2022.

Alp O, Erkip NK, Güllü R (2003) Outsourcing logistics: Designing transportation contracts between a manufacturer and a transporter. *Transportation Science* 37(1):23–39.

Benders JF (1962) Partitioning procedures for solving mixed-variables programming problems. *Numerische Mathematik* 4:238–252.

Bertazzi L, Bosco A, Laganà D (2015) Managing stochastic demand in an inventory routing problem with transportation procurement. *Omega* 56:112–121.

Bertazzi L, Maggioni F (2018) A stochastic multi-stage fixed charge transportation problem: Worst-case analysis of the rolling horizon approach. *European Journal of Operational Research* 267(2):555–569.

Bertazzi L, Speranza MG, Ukovich W (2000) Exact and heuristic solutions for a shipment problem with given frequencies. *Management Science* 46(7):973–988.

BIMCO (2022) BIMCO Contracts. https://www.bimco.org/contracts-and-clauses/bimco-contracts, Accessed March 1, 2022.

Birge JR (1985) Decomposition and partitioning methods for multistage stochastic linear programs. *Operations Research* 33(5):989–1007.

Birge JR, Louveaux F (2011) *Introduction to Stochastic Programming*, Springer, New York.

Boujemaa R, Rekik M, Hajji A (2022) An SDDP-based solution approach for carriers' selection and shipments assignment under dynamic stochastic demand. Technical Report CIRRELT-2022-17, Université Laval, Canada.

Caplice C, Sheffi Y (2003) Optimization-based procurement for transportation services. *Journal of Business Logistics* 24(2):109–128.

Caplice C, Sheffi Y, et al. (2005) Combinatorial auctions for truckload transportation. Cramton P, Shoham Y, Steinberg R, eds., *Combinatorial Auctions*, chapter 21, 539–571, MIT Press, Cambridge.

Cavagnini R, Bertazzi L, Maggioni F (2022) A rolling horizon approach for a multi-stage stochastic fixed-charge transportation problem with transshipment. *European Journal of Operational Research* 301(3):912–922.

Comtois C, Slack B (2016) Dynamic determinants in global iron ore supply chain. Technical Report CIRRELT-2016-06, Université de Montréal, Canada.

Garey MR, Johnson DS (1983) *Computers and intractability: A guide to the theory of NP-completeness*, W.H. Freeman, San Francisco.

Hu Q, Zhang Z, Lim A (2016) Transportation service procurement problem with transit time. *Transportation Research Part B: Methodological* 86:19–36.

Hu S, Han C, Dong ZS, Meng L (2019) A multi-stage stochastic programming model for relief distribution considering the state of road network. *Transportation Research Part B: Methodological* 123:64–87.

Lee CY, Shu S, Xu Z (2021) Optimal global liner service procurement by utilizing liner service schedules. *Production and Operations Management* 30(3):703–714.

Lim A, Qin H, Xu Z (2012) The freight allocation problem with lane cost balancing constraint. *European Journal of Operational Research* 217(1):26–35.

Lim A, Rodrigues B, Xu Z (2008) Transportation procurement with seasonally varying shipper demand and volume guarantees. *Operations Research* 56(3):758–771.

Lim A, Wang F, Xu Z (2006) A transportation problem with minimum quantity commitment. *Transportation Science* 40(1):117–129.

Lu T, Fransoo JC, Lee CY (2017) Carrier portfolio management for shipping seasonal products. *Operations Research* 65(5):1250–1266.

Ma Z, Kwon RH, Lee CG (2010) A stochastic programming winner determination model for truckload procurement under shipment uncertainty. *Transportation Research Part E: Logistics and Transportation Review* 46(1):49–60.

Meng Q, Hei X, Wang S, Mao H (2015) Carrying capacity procurement of rail and shipping services for automobile delivery with uncertain demand. *Transportation Research Part E: Logistics and Transportation Review* 82:38–54.

Page P (2021) Container shipping prices skyrocket as rush to move goods picks up. `https://www.wsj.com/articles/container-ship-prices-skyrocket-as-rush-to-move-goods-picks-up-11625482800`, Accessed March 1, 2022.

Papageorgiou DJ, Nemhauser GL, Sokol J, Cheon MS, Keha AB (2014) MIRPLib–A library of maritime inventory routing problem instances: Survey, core model, and benchmark results. *European Journal of Operational Research* 235(2):350–366.

Pereira MV, Pinto LM (1991) Multi-stage stochastic optimization applied to energy planning. *Mathematical Programming* 52(1):359–375.

Quezada F, Gicquel C, Kedad-Sidhoum S (2022) A stochastic dual dynamic integer programming based approach for remanufacturing planning under uncertainty. *International Journal of Production Research* Published online.

Remli N, Amrouss A, El Hallaoui I, Rekik M (2019) A robust optimization approach for the winner determination problem with uncertainty on shipment volumes and carriers' capacity. *Transportation Research Part B: Methodological* 123:127–148.

Remli N, Rekik M (2013) A robust winner determination problem for combinatorial transportation auctions under uncertain shipment volumes. *Transportation Research Part C: Emerging Technologies* 35:204–217.

Shapiro A (2011) Analysis of stochastic dual dynamic programming method. *European Journal of Operational Research* 209(1):63–72.

Shapiro A, Tekaya W, da Costa JP, Soares MP (2013) Risk neutral and risk averse stochastic dual dynamic programming method. *European Journal of Operational Research* 224(2):375–391.

Sheffi Y (2004) Combinatorial auctions in the procurement of transportation services. *Interfaces* 34(4):245–252.

Stecke KE, Zhao X (2007) Production and transportation integration for a make-to-order manufacturing company with a commit-to-delivery business mode. *Manufacturing & Service Operations Management* 9(2):206–224.

Thevenin S, Adulyasak Y, Cordeau JF (2022) Stochastic dual dynamic programming for multiechelon lot sizing with component substitution. *INFORMS Journal on Computing* Published online.

Valladão D, Silva T, Poggi M (2019) Time-consistent risk-constrained dynamic portfolio optimization with transactional costs and time-dependent returns. *Annals of Operations Research* 282(1):379–405.

Wu L, Wang S, Laporte G (2021) The robust bulk ship routing problem with batched cargo selection. *Transportation Research Part B: Methodological* 143:124–159.

Zhang B, Ding H, Li H, Wang W, Yao T (2014) A sampling-based stochastic winner determination model for truckload service procurement. *Networks and Spatial Economics* 14(2):159–181.

Zhang B, Yao T, Friesz TL, Sun Y (2015) A tractable two-stage robust winner determination model for truckload service procurement via combinatorial auctions. *Transportation Research Part B: Methodological* 78:16–31.

Appendix A: Mathematical Proofs

This section presents the proofs to the theorems, propositions, and lemmas introduced in the main text.

A.1. Proof of Theorem 1

Proof of Theorem 1. We only need to prove the case with a single scenario and a single period (i.e., $|\Xi|=1$ and $|\mathcal{T}|=1$), to which any case with $|\Xi|\geq 1$ and $|\mathcal{T}|\geq 1$ can be reduced. To prove its NP-hardness, we use a reduction from the following NP-complete problem (Garey and Johnson 1983).

Subset Sum Problem (SSP). Given a finite set $N=\{1,...,n\}$, size $d_i\in\mathbb{Z}^+$, $\forall i\in N$, and a positive integer B, is there a subset $N'\subseteq N$ such that $\sum_{i\in N'} d_i = B$? We only consider the case with $\sum_{i\in N} d_i > B$, as otherwise, the problem is trivial.

For any arbitrary instance of SSP with $\sum_{i\in N} d_i > B$, consider the following polynomial reduction to an instance of the SFPTMP with $|\Xi|=1$ and $|\mathcal{T}|=1$. Let each element $i\in N$ indicate a supply site with supply d_i, then we have $\mathcal{I}^S=\{1,...,n\}$. Let $\mathcal{I}^D=\{D_1,D_2\}$ be the set of demand sites with demand $d_{D_1}=-B$ and $d_{D_2}=B-\sum_{i\in\mathcal{I}^S} d_i$. We set the initial inventory levels $q_i^0=0$, $\forall i\in\mathcal{I}$. The upper bounds for holding inventories are set as $\bar{q}_i=0$, $\forall i\in\mathcal{I}$.

Let $\mathcal{L}=\{(i,j)|i\in\mathcal{I}^S, j\in\mathcal{I}^D\}$ be the set of lanes, and each lane $(i,j)\in\mathcal{L}$ has a shipment time $o_{i,j}=0$ (i.e., shipments can be completed within one period). Lane $(i,j)\in\mathcal{L}$ is associated with only one bid and let $\mathcal{B}_{(i,j)}=\{b_{(i,j)}\}$. The capacity range of any bid $b_{(i,j)}$ is set as $[d_i,d_i]$, $\forall (i,j)\in\mathcal{L}$. Further, the bid $b_{(i,j)}$ of any lane $(i,j)\in\mathcal{L}$ contains one shipment $r_{(i,j)}$ such that $t_1(r_{(i,j)})=t_2(r_{(i,j)})=1$. For any lane $(i,j)\in\mathcal{L}$, we set the freight rate for purchasing capacity from the bid $b_{(i,j)}$ as $f_{b_{(i,j)}}=1/d_i$. Besides, the variable shipping costs in the bids are set to be $g_{b_{(i,j)}}=0$, $\forall (i,j)\in\mathcal{L}$. In addition, for any site $i\in\mathcal{I}$, we set the unit inventory holding cost and the unit backlog cost to be $h_i=2n$ and $e_i=3n$, respectively. Finally, the standard freight rates are set as $c_{i,j}=2n$, $\forall (i,j)\in\mathcal{L}$. Now we prove that the minimum total cost of the instance is at most n if and only if the answer to the SSP is "yes".

On the one hand, suppose there exists such a subset N' for the SSP. Let $x_b\in\{0,1\}$ denote whether a bid $b\in\mathcal{B}$ is selected in the solution of the SFPTMP instance. Then for any $i\in N'$, we select bid $b_{(i,D_1)}$ (i.e., $x_{b_{(i,D_1)}}=1$) and set its capacity to be d_i. Meanwhile for any $i\in N\setminus N'$, we select bid $b_{(i,D_2)}$ (i.e., $x_{b_{(i,D_2)}}=1$) and set its capacity to be d_i. Let $z^1_{(i,j)}$ and $z^2_{(i,j)}$ denote the volume of the commodity shipped on lane $(i,j)\in\mathcal{L}$ through the shipment $r_{(i,j)}$ in bid $b_{(i,j)}$ and the standard freight rate, respectively. We then set $z^1_{(i,j)}=d_i$, if $x_{b_{(i,j)}}=1$ and $z^1_{(i,j)}=0$, otherwise, $\forall (i,j)\in\mathcal{L}$. Meanwhile, we let $z^2_{(i,j)}=0$, $\forall (i,j)\in\mathcal{L}$. Because $\sum_{i\in N'} d_i=-d_{D_1}$, $\sum_{i\in N\setminus N'} d_i=-d_{D_2}$, and $\sum_{i\in N} d_i=-(d_{D_1}+d_{D_2})$, the demand at each demand site is exactly satisfied with zero inventories and backlogs at all sites. Therefore, this is a feasible solution. From the settings of the cost components, the total cost is n, indicating that the instance has a minimum total cost no larger than n.

On the other hand, suppose that we have an optimal solution to the SFPTMP instance with a total cost no larger than n. Consider the subset $N'=\{i\in\mathcal{I}^S|x_{b_{(i,D_1)}}=1\}$. We next prove that $\sum_{i\in N'} d_i=B$, indicating that the answer to the SSP is "yes". To this end, we first show that for any supply site $i\in\mathcal{I}^S$, the equation $x_{b_{(i,D_1)}}+x_{b_{(i,D_2)}}=1$ must hold in the optimal solution. We prove this by contradiction.

First, suppose that in the optimal solution, there exists a supply site $i' \in \mathcal{I}^S$ such that $x_{b_{(i',D_1)}} + x_{b_{(i',D_2)}} = 0$. In this case, let $\delta_{i'}$ denote the remaining (unshipped) supply at this site, indicating that a total volume of $(d_{i'} - \delta_{i'})$ is shipped from this site through the standard rates. As a result, the shipping cost of this solution should at least be $2n(d_{i'} - \delta_{i'})$. In addition, because $q_i^0 = 0$ and $\bar{q}_i = 0$ with $i \in \mathcal{I}$, the backlog level at site i' equals $\delta_{i'}$. Further, because $\sum_{i\in\mathcal{I}^S} d_i = -(d_{D_1} + d_{D_2})$, the total backlog volume at the demand sites must also at least be $\delta_{i'}$. These indicate that the backlog cost of the solution is at least $6n\delta_{i'}$. Summing these costs, we have that the total cost of this solution should at least be $2nd_{i'} + 4n\delta_{i'} > n$, which forms a contradiction. Therefore, the equation $x_{b_{(i,D_1)}} + x_{b_{(i,D_2)}} \geq 1$ must hold for any $i \in \mathcal{I}^S$ in the optimal solution.

Second, suppose there exists a supply site $i' \in \mathcal{I}^S$ such that $x_{b_{(i',D_1)}} + x_{b_{(i',D_2)}} = 2$. For any lane $(i,j) \in \mathcal{L}$, bid $b_{(i,j)}$ is associated with a freight rate $1/d_i$ and a capacity range $[d_i, d_i]$, which indicates that the cost of selecting any bid equals 1. Therefore, the total cost of selecting bids in the solution can be calculated as $\sum_{i\in\mathcal{I}^S\setminus\{i'\}}(x_{b_{(i,D_1)}} + x_{b_{(i,D_2)}}) + x_{b_{(i',D_1)}} + x_{b_{(i',D_2)}}$. Since we have proved that $x_{b_{(i,D_1)}} + x_{b_{(i,D_2)}} \geq 1$, $\forall i \in \mathcal{I}^S$, if $x_{b_{(i',D_1)}} + x_{b_{(i',D_2)}} = 2$, the cost of bid selection in this solution should at least be $n+1$, which again forms a contradiction. Therefore, we have $x_{b_{(i,D_1)}} + x_{b_{(i,D_2)}} = 1$, for any $i \in \mathcal{I}^S$ in the optimal solution.

Because $x_{b_{(i,D_1)}} + x_{b_{(i,D_2)}} = 1$, $\forall i \in \mathcal{I}^S$, the cost for bid selection in the solution equals n. This further indicates that the inventory and backlog levels at all sites should be zero and the volume of the shipment acquired through the standard freight rate at any lane should also be zero in the solution. For these conditions to hold, the solution must satisfy $\sum_{i\in N'} d_i = B$. This completes the proof. □

A.2. Proof of Proposition 1

Let $\boldsymbol{x}^* = (x_b^* | b \in \mathcal{B})$, $\boldsymbol{y}^* = (y_b^* | b \in \mathcal{B})$, $\boldsymbol{z}^* = (z_{\xi,a}^* | a \in \mathcal{A}, \xi \in \Xi)$, $\boldsymbol{u}^* = (u_{\xi,n}^* | n \in \mathcal{N}, \xi \in \Xi)$, and $\boldsymbol{v}^* = (v_{\xi,n}^* | n \in \mathcal{N}, \xi \in \Xi)$ be the vectors for the values of variables x_b, y_b, $z_{\xi,a}$, $u_{\xi,n}$, and $v_{\xi,n}$ in an *optimal* solution (denoted by $\boldsymbol{X}^*$) of $\mathbf{P}$. We have the following lemma.

LEMMA A.1. *$\boldsymbol{X}^*$ satisfies the following equalities:*

$$\min\{v_{\xi,n}^*, \overline{q}_n - u_{\xi,n}^*\} = 0 \;\; \forall n \in \mathcal{N}^S, \forall \xi \in \Xi, \tag{A.1}$$

$$\min\{u_{\xi,n}^*, v_{\xi,n}^*\} = 0 \;\; \forall n \in \mathcal{N}^D, \forall \xi \in \Xi. \tag{A.2}$$

Proof of Lemma A.1. Supposing (A.1) do not hold, for some $n \in \mathcal{N}^S$, we must have $u_{\xi,n}^* < \overline{q}_n$ and $v_{\xi,n}^* > 0$. Let $\sigma = \min\{v_{\xi,n}^*, \overline{q}_n - u_{\xi,n}^*\}$. We have $\sigma > 0$.

Consider a solution (denoted by $\boldsymbol{X}'$) for problem $\mathbf{P}$ in which $u_{\xi,n} = u_{\xi,n}^* + \sigma$ and $v_{\xi,n} = v_{\xi,n}^* - \sigma$ and other variables remain the same as in $\boldsymbol{X}^*$. It is easy to check that $\boldsymbol{X}'$ is feasible. Let Z' and Z^* denote objective function values associated with $\boldsymbol{X}'$ and $\boldsymbol{X}^*$, respectively. We have $Z' - Z^* = (h_n - e_n)\sigma$. Because $h_n < e_n$, $Z' - Z^* < 0$, which is a contradiction of the optimality of $\boldsymbol{X}^*$. Therefore, (A.1) must hold for any optimal solution of $\mathbf{P}$.

The process to show that (A.2) must hold for any optimal solution of $\mathbf{P}$ is similar, and thus we omit it here. □

Proof of Proposition 1. From the definition of Λ_p, we have $(\xi_1,\xi_2)\in\Lambda_p$ if and only if $(\xi_1,\xi_2)\in\Lambda_{p'}$, $\forall p'\in\{1,...,p\}$, where $p\in\mathcal{P}$. Then, given any $(\xi_1,\xi_2)\in\Lambda_p$ and $p\in\mathcal{P}$, due to constraints (10), one must have

$$z^*_{\xi_1,a}=z^*_{\xi_2,a}\ \ \forall a\in\mathcal{A}_{p'},\forall p'\in\{1,...,p\}. \tag{A.3}$$

It is therefore easy to infer that

$$\sum_{a\in A^+(n)} z^*_{\xi_1,a}=\sum_{a\in A^+(n)} z^*_{\xi_2,a}\ \ \forall n\in\mathcal{N}_p,\forall p'\in\{1,...,p\}, \tag{A.4}$$

$$\sum_{a\in A^-(n)} z^*_{\xi_1,a}=\sum_{a\in A^-(n)} z^*_{\xi_2,a}\ \ \forall n\in\mathcal{N}_p,\forall p'\in\{1,...,p\}. \tag{A.5}$$

Further, combining these two equations with constraints (5)–(8) gives us:

$$u^*_{\xi_1,n}+v^*_{\xi_1,n}=u^*_{\xi_2,n}+v^*_{\xi_2,n}\ \ \forall n\in\mathcal{N}_p\cap\mathcal{N}^S,\forall p'\in\{1,...,p\}, \tag{A.6}$$

$$u^*_{\xi_1,n}-v^*_{\xi_1,n}=u^*_{\xi_2,n}-v^*_{\xi_2,n}\ \ \forall n\in\mathcal{N}_p\cap\mathcal{N}^D,\forall p'\in\{1,...,p\}. \tag{A.7}$$

Finally, from the results in Lemma A.1 and equations (A.6) and (A.7), it is easy to infer that

$$u^*_{\xi_1,n}=u^*_{\xi_2,n}\ \ \forall n\in\mathcal{N}_p,\forall p'\in\{1,...,p\}, \tag{A.8}$$

$$v^*_{\xi_1,n}=v^*_{\xi_2,n}\ \ \forall n\in\mathcal{N}_p,\forall p'\in\{1,...,p\}. \tag{A.9}$$

Therefore, by solving $\mathbf{P}$ to optimality, we have identical decisions under scenarios $\xi_1,\xi_2\in\Lambda_p$ in any stage $p'\in\{1,...,p\}$. This completes the proof. □

A.3. Proof of Lemma 1

Proof of Lemma 1. Given any site $i\in\mathcal{I}^D$, let $\bar{\omega}^i_p=\arg\max_{\omega\in\Omega_p}\sum_{t=\underline{t}_p}^{\bar{t}_p} d^{\omega}_{i,t}$, and let $\bar{\omega}^i_{p,t}=\arg\max_{\omega\in\Omega_p}\sum_{t'=\underline{t}_p}^{t} d^{\omega}_{i,t'}$, where $t\in\mathcal{T}_p$ and $p\in\mathcal{P}$.

Given any stages $p_1,p_2\in\mathcal{P}$ with $p_2>p_1$ and a period $t_2\in\mathcal{T}_{p_2}$, for any $i\in\mathcal{I}^D$, let Ξ^0 be the set of scenarios such that $\forall\xi\in\Xi^0$, $\omega_p(\xi)=\bar{\omega}^i_p$, $\forall p\in\{p_1+1,...,p_2-1\}$ and $\omega_{p_2}(\xi)=\bar{\omega}^i_{p_2,t_2}$. By summing constraints (7) for site i under any scenario $\xi^0\in\Xi^0$ over all periods $t'\in\{\underline{t}_{p_1+1},...,t_2\}$ we have

$$u_{\xi^0,n_2}-v_{\xi^0,n_2}=u_{\xi^0,n_1}-v_{\xi^0,n_1}+\bar{d}_{i,p_1+1,t_2}+\sum_{t'=\underline{t}_{p_1+1}}^{t_2}\sum_{n=(i,t')\in\mathcal{N}}\sum_{a\in A^-(n)} z_{\xi^0,a}\ \ \forall\xi^0\in\Xi^0, \tag{A.10}$$

where $n_1=(i,\bar{t}_{p_1})$ and $n_2=(i,t_2)$.

Because $v_{\xi^0,n_2}\geq 0$ and $z_{\xi^0,a}\geq 0$, $\forall a\in A^-(n)$ the following inequality holds:

$$u_{\xi^0,n_2}\geq u_{\xi^0,n_1}-v_{\xi^0,n_1}+\bar{d}_{i,p_1+1,t_2}+\sum_{t'=\underline{t}_{p_1+1}}^{t_2}\sum_{n=(i,t')\in\mathcal{N}}\sum_{p'=1}^{p_1}\sum_{a\in A^-(n)\cap\mathcal{A}_{p'}} z_{\xi^0,a}\ \ \forall\xi^0\in\Xi^0. \tag{A.11}$$

Due to constraints (9), we have

$$u_{\xi^0,n_1}-v_{\xi^0,n_1}+\bar{d}_{i,p_1+1,t_2}+\sum_{t'=\underline{t}_{p_1+1}}^{t}\sum_{n=(i,t')\in\mathcal{N}}\sum_{p'=1}^{p_1}\sum_{a\in A^-(n)\cap\mathcal{A}_{p'}} z_{\xi^0,a}\leq\bar{q}_{n_2}\ \ \forall\xi^0\in\Xi^0. \tag{A.12}$$

Further, given any $\xi^0\in\Xi^0$, let $\Xi(\xi^0)\subseteq\Xi$ be the set of scenarios such that $\Xi(\xi^0)=\{\xi\in\Xi|\xi=\xi^0\vee(\xi,\xi^0)\in\Lambda_{p_1}\}$. From Proposition 1, we have

$$u_{\xi,n_1}-v_{\xi,n_1}+\bar{d}_{i,p_1+1,t_2}+\sum_{t'=\underline{t}_{p+1}}^{t}\sum_{n=(i,t')\in\mathcal{N}}\sum_{p'=1}^{p_1}\sum_{a\in A^-(n)\cap\mathcal{A}_{p'}} z_{\xi,a}\leq\bar{q}_{n_2}\ \ \forall\xi\in\Xi(\xi^0),\forall\xi^0\in\Xi^0. \tag{A.13}$$

In addition, the structure of the scenario tree implies that $\bigcup_{\xi^0\in\Xi^0}\Xi(\xi^0)=\Xi$, and the final result follows directly. □

A.4. Proof of Proposition 2

Proof of Proposition 2. We show that problem $\mathbf{P}'_{\xi,p}$, where $p \in \mathcal{P}$ and $\xi \in \Xi$, is feasible by constructing a feasible solution ($\mathcal{S}$) to the problem as follows.

First, in $\mathcal{S}$, we let $z_{\xi,a} = 0$, $\forall a \in \mathcal{A}_p$. In the sequel, for the solution to be feasible, we must have

$$u_{\xi,n_2} = \bar{u}_{\xi,p-1,n_1} + \bar{v}_{\xi,p-1,n_1} + \sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} d_{\xi,n} - v_{\xi,n_2}$$
$$\forall n_1 = (j, \bar{t}_{p-1}), n_2 = (j,t) \in \mathcal{N}, \forall j \in \mathcal{I}^S, \forall t \in \mathcal{T}_p, \tag{A.14}$$
$$u_{\xi,n_2} = \bar{u}_{\xi,p-1,n_1} - \bar{v}_{\xi,p-1,n_1} + \sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} (d_{\xi,n} + \sum_{a\in A^-(n)\cap\widetilde{\mathcal{A}}_{p-1}} \bar{z}_{\xi,p-1,a}) + v_{\xi,n_2}$$
$$\forall n_1 = (i, \bar{t}_{p-1}), n_2 = (j,t) \in \mathcal{N}, \forall j \in \mathcal{I}^D, \forall t \in \mathcal{T}_p. \tag{A.15}$$

To show that such a feasible $\mathcal{S}$ exists, it suffices to show that for any $n_2 = (j,t) \in \mathcal{N}_p$ there exists a $v_{\xi,n_2} \geq 0$ such that:

$$\bar{u}_{\xi,p-1,n_1} + \bar{v}_{\xi,p-1,n_1} + \sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} d_{\xi,n} - v_{\xi,n_2} \geq 0, \tag{A.16}$$
$$\bar{u}_{\xi,p-1,n_1} + \bar{v}_{\xi,p-1,n_1} + \sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} d_{\xi,n} - v_{\xi,n_2} \leq \bar{q}_{n_2}, \tag{A.17}$$

if $j \in \mathcal{I}^S$ and

$$\bar{u}_{\xi,p-1,n_1} - \bar{v}_{\xi,p-1,n_1} + \sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} (d_{\xi,n} + \sum_{a\in A^-(n)\cap\widetilde{\mathcal{A}}_{p-1}} \bar{z}_{\xi,p-1,a}) + v_{\xi,n_2} \geq 0, \tag{A.18}$$
$$\bar{u}_{\xi,p-1,n_1} - \bar{v}_{\xi,p-1,n_1} + \sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} (d_{\xi,n} + \sum_{a\in A^-(n)\cap\widetilde{\mathcal{A}}_{p-1}} \bar{z}_{\xi,p-1,a}) + v_{\xi,n_2} \leq \bar{q}_{n_2}, \tag{A.19}$$

if $j \in \mathcal{I}^D$, where $n_1 = (j, \bar{t}_{p-1}) \in \mathcal{N}$.

One can easily verify that inequalities (A.16)–(A.19) hold as long as we have:

$$\bar{u}_{\xi,p-1,n_1} - \bar{v}_{\xi,p-1,n_1} + \sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} (d_{\xi,n} + \sum_{a\in A^-(n)\cap\widetilde{\mathcal{A}}_{p-1}} \bar{z}_{\xi,p-1,a}) \leq \bar{q}_{n_2} \tag{A.20}$$

for the case $j \in \mathcal{I}^D$.

Note that $d_{\xi,n} \leq 0$, $\forall n \in \mathcal{N}^D$ and $\bar{q}_{n_2} = \bar{q}_j$, $\forall n_2 = (j,t) \in \mathcal{N}^D$. Hence, if $p = 1$, we have $\widetilde{\mathcal{A}}_0 = \varnothing$ and (A.20) holds directly as long as the original problem $\mathbf{P}$ is feasible. If $p > 1$, (A.20) holds if we have:

$$\bar{u}_{\xi,p-1,n_1} - \bar{v}_{\xi,p-1,n_1} + \sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} \sum_{a\in A^-(n)\cap\widetilde{\mathcal{A}}_{p-1}} \bar{z}_{\xi,p-1,a} + \sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} d_{\xi,n} \leq \bar{q}_{n_2}$$
$$\forall n_1 = (i, \bar{t}_{p-1}), n_2 = (j, t_2) \in \mathcal{N}^D, \forall ((i,t_1),(j,t_2)) \in \widetilde{\mathcal{A}}_{p-1}. \tag{A.21}$$

By definition, we have $\sum_{t'=\underline{t}_p}^{t} \sum_{n=(j,t')\in\mathcal{N}} d_{\xi,n} \leq \bar{d}_{j,p,t}$. Therefore, from constraints (45), we have that (A.20) is valid for $\mathbf{P}'_{\xi,p}$ with $p > 1$. This completes the proof. □

A.5. Proof of Proposition 3

The proof is similar to that of Proposition 2 and is thus omitted here.

A.6. Proof of Proposition 4

Proof of Proposition 4. Given any feasible solution $(\boldsymbol{x}, \boldsymbol{y})$ from stage 0, consider the problems $\mathbf{R}(\boldsymbol{x}, \boldsymbol{y})$ and $\widehat{\mathbf{R}}(\boldsymbol{x}, \boldsymbol{y})$ which are formulated as follows:

$$\begin{aligned} \mathbf{R}(\boldsymbol{x}, \boldsymbol{y}) \;&=\; \min \sum_{\xi \in \Xi} \rho_\xi \left(\sum_{n \in \mathcal{N}} (h_n u_{\xi,n} + e_n v_{\xi,n}) + \sum_{a \in \mathcal{A}} c_a z_{\xi,a} \right) \\ &\text{s.t. } (4) - (10), (13) - (15), \end{aligned} \tag{A.22}$$

$$\begin{aligned} \widehat{\mathbf{R}}(\boldsymbol{x}, \boldsymbol{y}) \;&=\; \min \sum_{\widehat{\xi} \in \widehat{\Xi}_0} \widehat{\rho}_{\widehat{\xi}} \left(\sum_{n \in \mathcal{N}} \left(h_n \widehat{u}_{\widehat{\xi},n} + e_n \widehat{v}_{\widehat{\xi},n}\right) + \sum_{a \in \mathcal{A}} c_a \widehat{z}_{\widehat{\xi},a} \right) \\ &\text{s.t. } (51) - (61). \end{aligned} \tag{A.23}$$

One can easily verify that these problems are feasible and bounded. Let Z_1 and Z_2 denote the optimal objective function values of $\mathbf{R}$ and $\widehat{\mathbf{R}}$, respectively. Then, because of Theorem 1 in Chapter 10 of Birge and Louveaux (2011), we have $Z_2 \leq Z_1$. The validity of the optimality inequalities (50)–(61) follows directly from the result. □

A.7. Proof of Proposition 5

The proof is similar to that of Proposition 4 and is thus omitted here.

Appendix B: The SDDP Framework

This section presents the pseudocode of the SDDP approach.

Algorithm B.1 Stochastic Dual Dynamic Programming (SDDP).

Initialize: $LB \leftarrow -\infty$, $l \leftarrow 1$, and initial cost-to-go functions Ψ_p, $p \in \mathcal{P}^+$
while Some stopping criterion is not satisfied **do**
    /*Sampling step*/
    Sample a set of scenarios $\Xi^l \subseteq \Xi$
    /*Forward step*/
    Solve the problem $\mathbf{P}_0(\Psi_0)$
    Collect $\bar{\boldsymbol{\chi}}_{\xi,0}$, $\forall \xi \in \Xi^l$
    Set LB equal to the optimal value of $\mathbf{P}_0(\Psi_0)$
    for $\xi \in \Xi^l$ **do**
        for $p = 1, ..., \bar{p}$ **do**
            Solve the problem $\mathbf{P}_{\xi,p}(\bar{\boldsymbol{\chi}}_{\xi,p-1}, \Psi_p)$
            Collect $\bar{\boldsymbol{\chi}}_{\xi,p}$
        end for
    end for
    /*Backward step*/
    for $\xi \in \Xi^l$ **do**
        for $p = \bar{p}, ..., 1$ **do**
            for $\omega \in \Omega_p$ **do**
                Solve the dual of problem $\mathbf{Q}_{\xi,\omega,p}(\bar{\boldsymbol{\chi}}_{\xi,p-1}, \Psi_p)$
            end for
            Update Ψ_{p-1} by adding valid cuts
        end for
    end for
    $l \leftarrow l + 1$
end while

Appendix C: Details of Instance Generation

In this section, we first introduce the methods for creating the two types of cases and then explain how the supply data were generated in the instances.

C.1. Generation of Type-I Cases

The type-I cases were generated based on five instances selected from the instance set provided by Papageorgiou et al. (2014) for the maritime inventory-routing problem (MIRP). In each of the five selected MIRP instances, there are one supply port and eight demand ports. Each instance covers a planning horizon of 360 days and the (deterministic) daily supply or demand generated at each port is provided.

We proceed as follows to convert an MIRP instance into a type-I case. To begin with, each supply (demand) port in the MIRP instance corresponds to a supply (demand) site in a case. Second, in each case, we let a period $t \in \mathcal{T}$ contain seven consecutive days (a week) and the planning horizon consists of 54 periods (378 days). The nominal demand $\bar{d}_{it}$ at site $i \in \mathcal{I}^D$ in period $t \in \mathcal{T}$ is set equal to the sum of the daily demands of the corresponding port that are associated with period t in the MIRP instance (we set the daily demands of the days later than the 360th day equal to those of the 360th day in the MIRP instance).

Other parameters in a case were generated as follows. We set the initial inventory (q_i^0) and the maximum inventory level ($\bar{q}_i$) at each site $i \in \mathcal{I}$ to the same values as those of the corresponding port in the associated MIRP instance. We let the unit inventory cost $h_i = 0$, $\forall i \in \mathcal{I}$, which is consistent with the setting in the MIRP instances. For any site $i \in \mathcal{I}$, its unit backlogging cost is set as $e_i = 0.05$, if $i \in \mathcal{I}^S$ and $e_i = 1.1(\max_{j \in \mathcal{I}^S} c_{j,i})$, if $i \in \mathcal{I}^D$, where $c_{j,i}$ is the standard freight rate on lane $(j,i) \in \mathcal{L}$.

The commodity can be shipped on the lane between any supply site and any demand site. We let the shipping time $o_{i,j} = \lceil \bar{o}_{i,j}/7 \rceil$ for all $(i,j) \in \mathcal{L}$, where $\bar{o}_{i,j}$ (in days) is the travel time between the corresponding ports in the associated MIRP instance. The standard freight rate on each lane $(i,j) \in \mathcal{L}$ is set as $c_{i,j} = 0.0005 DIS_{i,j}$, where $DIS_{i,j}$ represents the distance (km) between the corresponding ports in the original MIRP instance.

The bids were created as follows. Each bid is characterized by a shipment capacity range and a shipment frequency. Let $\overline{C}$ be the maximum of the vessel capacities in the original MIRP instance, and let $\overline{Q}_j = \min\{\overline{C}, \bar{q}_j\}$, $\forall j \in \mathcal{I}$. For generating the bids on a lane $(i,j) \in \mathcal{L}$, three shipment capacity ranges were used, which are $[\lfloor 0.25\overline{Q}_j \rfloor, \lfloor 0.5\overline{Q}_j \rfloor]$, $[\lfloor 0.5\overline{Q}_j \rfloor + 1, \lfloor 0.75\overline{Q}_j \rfloor]$, and $[\lfloor 0.75\overline{Q}_j \rfloor + 1, \lfloor \overline{Q}_j \rfloor]$. We also used three shipping frequencies, where the intervals between two consecutive shipments in a bid are set to two, four, and six periods. There are thus nine combinations of capacity ranges and shipping frequencies, and for each combination, we generate a bid. Hence, we have $|\mathcal{B}_{i,j}| = 9$, $\forall (i,j) \in \mathcal{L}$. The freight rate f_b of a bid $b \in \mathcal{B}_{i,j}$ was set as follows:

$$f_b = \begin{cases} 0.8c_{i,j}, \text{ if } [\underline{m}_b, \overline{m}_b] = [\lfloor 0.25\overline{Q}_j \rfloor, \lfloor 0.5\overline{Q}_j \rfloor], \\ 0.7c_{i,j}, \text{ if } [\underline{m}_b, \overline{m}_b] = [\lfloor 0.5\overline{Q}_j \rfloor + 1, \lfloor 0.75\overline{Q}_j \rfloor], \\ 0.6c_{i,j}, \text{ if } [\underline{m}_b, \overline{m}_b] = [\lfloor 0.75\overline{Q}_j \rfloor + 1, \lfloor \overline{Q}_j \rfloor]. \end{cases}$$

Further, the variable transportation cost in each bid $b \in \mathcal{B}$ was set as $g_b = 0$. Moreover, as for the shipment schedules, given any bid b, the start time of its first shipment was randomly selected from the set of periods $\{1, 2, 3\}$ and the start times of subsequent shipments were set according to the shipping frequency. The number of shipments in the bid was set to the maximum number of shipments that can be completed within the planning horizon, which was determined by the start time of the first shipment, the shipping frequency, and the transportation time of a shipment in the bid.

C.2. Generation of Type-II Cases

We use the manufacturing and iron ore transportation data from a large steel manufacturer in China and synthetic data to generate the five type-II cases.

The studied steel manufacturer has two plants (plants j_1 and j_2) in China. From 2015 to 2019, the manufacturer consumed some 4,700 to 5,300 thousand tonnes of iron ore each year. Iron ore is obtained from the suppliers (mines) in Australia (supplier i_1) and Brazil (supplier i_2). Iron ore from suppliers i_1 and i_2 is loaded at the port of Port Hedland in Australia and the port of Tubarao in Brazil, respectively, and iron ore for plants j_1 and j_2 is unloaded at the port of Tianjin and the port of Tangshan in China, respectively. We

Table C.1 Shipping Times and Standard Freight Rates.

Lane	Supplier	Plant	$o_{i,j}$ (days)	$c_{i,j}$ (USD per tonne)
1	i_1	j_1	14	25
2	i_1	j_2	14	25
3	i_2	j_1	42	55
4	i_2	j_2	42	55

thus have four lanes for shipping the iron ore. The shipping times and standard freight rates for these lanes were set as in Table C.1.

Iron ore can be stored at the yards of both the suppliers and the plants. Considering the large capacities of yards of the mines, we assume that there are infinite capacities for holding the iron ore at the supply sites. We also assume that the initial inventories of the iron ore supplies in the suppliers are zero. As for the capacities of the yards at plants j_1 and j_2, we have $\overline{q}_{j_1} = 200$ and $\overline{q}_{j_2} = 150$ thousand tonnes, respectively. We assume that the initial inventories in plants j_1 and j_2 are 100 and 75 thousand tonnes, respectively.

To promote fast delivery, the inventory holding costs at suppliers were set to be 0.1 USD per tonne per week. Since the yards in the plants are owned by the manufacturer, we set zero inventory holding costs at the plants. The backlogging costs were set as 0.2 and 60 USD per tonne per period at the suppliers and the plants, respectively.

Accordingly, each type-II case has the same distribution network as in the real situation. Further, to generate the bids for the capacity contracts on the lanes in a case, we used four shipment capacity ranges ($[\underline{m}_b, \overline{m}_b]$), including $[40, 65]$, $[66, 100]$, $[101, 150]$, and $[151, 200]$ thousand tonnes, which are in accordance with the capacities of Handymax bulk carriers, Panamax bulk carriers, and small and large Capesize bulk carriers, respectively. We also used four shipment frequencies, where the intervals between two consecutive shipments on a lane were two, four, six, and eight periods respectively. For each lane, we generated 16 bids, each corresponding to a combination of a capacity range and a frequency. The freight rate f_b of a bid $b \in \mathcal{B}_{i,j}$ was set as follows:

$$f_b = \begin{cases} 0.8c_{i,j}, \text{ if } [\underline{m}_b, \overline{m}_b] = [40, 65], \\ 0.7c_{i,j}, \text{ if } [\underline{m}_b, \overline{m}_b] = [66, 100], \\ 0.6c_{i,j}, \text{ if } [\underline{m}_b, \overline{m}_b] = [101, 150], \\ 0.5c_{i,j}, \text{ if } [\underline{m}_b, \overline{m}_b] = [151, 200]. \end{cases}$$

Other parameters regarding the bids were generated in the same way described in Appendix C.1.

In each case, the planning horizon contains 54 periods and each period represents a week. The nominal demands in each period in each case were generated by referring to the historical weekly iron ore consumption data of the manufacturer in a unique one-year period from 2015 to 2019.

C.3. Settings of Supplies in the Instances

In any SFPTMP instance, supplies are generated only in the first period ($\underline{t}_p$) in any stage $p \in \mathcal{P}$. That is, we let $d^{\omega}_{i,t} = 0$, $\forall t \in \mathcal{T}_p \setminus \{\underline{t}_p\}$, $\forall p \in \mathcal{P}$, $\forall i \in \mathcal{I}^S$.

For SFPTMP instances generated from type-I cases, we assume that supplies and demands are balanced in each stage. In particular, in any of these instances, given a stage $p \in \mathcal{P}$ and a stage scenario $\omega \in \Omega_p$, the supply produced in the (sole) supply site in period $\underline{t}_p$ under this scenario was set equal to $-\sum_{i \in \mathcal{I}^D} \sum_{t \in \mathcal{T}_p} d^{\omega}_{i,t}$.

Supplies in the instances generated from type-II cases were set to mimic the iron ore procurement strategy of the manufacturer. In particular, supplies of iron ore come from two sources: long-term purchase contracts (LPCs) and short-term agreements (SAs). The manufacturer imports approximately 1,200 and 900 thousand tonnes of iron ore through LPCs per year from suppliers i_1 and i_2, respectively. The volume in an LPC is evenly produced by the mine within the contractual period and SAs are used only when supplies from the LPCs cannot meet the demand in a certain period. While the manufacturer has LPCs with both suppliers, all SAs come from supplier i_1.

In this context, for an instance, we let $d^{lpc}_{i_1,p}$ and $d^{lpc}_{i_2,p}$ denote the supplies provided by the suppliers in stage $p \in \mathcal{P}$ for the LPCs and let $d^{sa,\omega}_p$ be the supplies from SAs under stage scenario $\omega \in \Omega_p$ in this stage. Then, we let $d^{\omega}_{i_1,\underline{t}_p} = d^{lpc}_{i_1,p} + d^{sa,\omega}_p$, where $d^{sa,\omega}_p = \max\{0, -\sum_{t \in \mathcal{T}_p}(d^{\omega}_{j_1,t} + d^{\omega}_{j_2,t}) - d^{lpc}_{i_1,p} - d^{lpc}_{i_2,p}\}$, and $d^{\omega}_{i_2,\underline{t}_p} = d^{lpc}_{i_2,p}$, where $\omega \in \Omega_p$ and $p \in \mathcal{P}$.